CHAPTER 5

PLANE THEORY OF ELASTICITY IN RECTANGULAR CARTESIAN COORDINATES

If a problem of elasticity is reducible to a two-dimensional problem, we say that it is a *plane problem of elasticity*. The corresponding theory is referred to as the *plane theory of elasticity*.

The equations of the plane theory of elasticity apply to the following two cases of equilibrium of elastic bodies, which are of considerable interest in practice: (1) *plane strain* (Section 5-1) and (2) *deformation of a thin plate under forces applied to its boundary and acting in its plane* (Section 5-2).

In the past decade or so, a considerable literature on the application of complex variables to the analytical solution of plane problems has evolved. In fact, the complex-variable method has been developed to the extent that it is currently considered a routine approach to the plane problem of elasticity. However, in many plane problems, the complex-variable method is now being superseded by numerical methods such as finite element methods, which lend themselves to the treatment of difficult boundary value problems in engineering. The complex-variable method has been expounded extensively and authoritatively by Muskhelishvili (1975) and also by Sokolnikoff (1983). Consequently, the method is treated only briefly in this book (see Appendix 5B).

In Appendix 5A we discuss briefly the problem of plane elasticity with couple stresses.

5-1 Plane Strain

The plane strain approximation, which serves to represent a three-dimensional problem by a two-dimensional one, may be applicable to a prismatic body whose length is large compared to its cross-sectional dimensions and which is loaded uniformly along its length. An example of such a body is a long hollow cylinder subjected to lateral pressure. In such bodies the longitudinal displacement component—say, w in the z-direction—is often very small compared to the displacement components in the cross section—say, in the (x, y) plane—and under certain conditions may be ignored. A formal definition of plane strain is given below, and the equations of elasticity are simplified accordingly.

For convenience, we employ (x, y, z) notation.

Definition. *A body is in a state of plane strain, parallel to the (x, y) plane, if the displacement w is zero, and if the components (u, v) are functions of (x, y) only.*

In view of this definition, the cubical strain for plane strain is

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{5-1.1}$$

Hence, Eqs. (4-6.5) reduce to (isotropic material) (see Table 3-2.1)

$$\begin{aligned} &\sigma_x = \lambda e + 2G\epsilon_x, \qquad \sigma_y = \lambda e + 2G\epsilon_y, \qquad \sigma_z = \lambda e \\ &\tau_{xy} = G\gamma_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \qquad \tau_{xz} = \tau_{yz} = 0 \end{aligned} \tag{5-1.2}$$

Equations (5-1.2) show that the stress components are functions of (x, y) only, because (u, v), hence e, are functions of (x, y) only.

The equilibrium equations for plane strain [see Eqs. (3-8.1)] are

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \\ &Z = 0 \end{aligned} \tag{5-1.3}$$

Consequently, in plane strain with respect to the (x, y) plane, the component of body force perpendicular to the (x, y) plane must vanish. Also, because σ_x, σ_y, τ_{xy} are functions of (x, y) only, the components (X, Y) of the body force are independent of z.

The strain–displacement relations [Eqs. (2-15.14)] reduce to the following form for plane strain:

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = 0$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{xz} = \gamma_{yz} = 0 \tag{5-1.4}$$

Hence, by Eqs. (4-6.6) and (5-1.4),

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{5-1.5}$$

Thus, the static equations of elasticity for a body in plane strain with respect to the (x, y) plane reduce to

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0$$
$$\sigma_x = \lambda e + 2G\epsilon_x, \qquad \sigma_y = \lambda e + 2G\epsilon_y, \tag{5-1.6}$$
$$\sigma_z = \lambda e = \nu(\sigma_x + \sigma_y), \qquad \tau_{xy} = G\gamma_{xy}$$

In Eqs. (5-1.6) it should be noted that σ_z is deduced from σ_x and σ_y [Eq. (5-1.5)]. Hence, *the problem is reduced to determining three stress components* σ_x, σ_y, τ_{xy}.

With Eq. (5-1.5), the stress–strain relations [Eqs. (4-6.8)] may be written in the form

$$\epsilon_x = \frac{1+\nu}{E}[\sigma_x - \nu(\sigma_x + \sigma_y)]$$
$$\epsilon_y = \frac{1+\nu}{E}[\sigma_y - \nu(\sigma_x + \sigma_y)] \tag{5-1.7}$$
$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

A state of plane strain can be maintained in a cylindrically shaped body by suitably applied forces. For example, by Eq. (5-1.5), we see that σ_z does not vanish in general. Hence, for a state of plane strain in a cylindrical body with the generators of the body parallel to the z axis, a tension or compression σ_z must be applied over the terminal sections formed by planes perpendicular to the z axis. Thus, the effect of σ_z is to keep constant the length of all longitudinal fibers of the body. In addition, the stress components σ_x and σ_y must attain values on the lateral surface of the body that are consistent with Eqs. (5-1.2) or Eqs. (5-1.6).

The solution of the plane-strain problem of the cylindrical body may be used in conjunction with the auxiliary problem of a cylindrical body subjected to longitudinal terminal forces to solve the problem of deformation of a cylindrical body with terminal sections free of force. If the longitudinal terminal forces are equal in magnitude but opposite in sign to σ_z, the superposition of the results clears the terminal sections of the cylinder of force. However, the resulting deformation of the body is not necessarily a plane deformation. In general, the solution of the auxiliary problem involves the deformation of a cylinder by longitudinal end forces that produce a net axial force and a net couple (pure bending); see Chapter 7.

Example 5-1.1. Plane State of Strain. A region in the (x, y) plane is subjected to a state of plane strain such that the (x, y) displacement components (u, v) are linear functions of (x, y), namely (with $w = 0$ in the z direction)

$$u = a_1 x + b_1 y + c_1, \qquad v = a_2 x + b_2 y + c_2, \qquad w = 0 \tag{a}$$

Measurements indicate that for $x = 0$, $y = 1$ m, $u = -3$ mm, $v = 2.5$ mm; for $x = 1$ m, $y = 0$, $u = -2$ mm, $v = 1$ mm; for $x = 1$ m, $y = 1$ m, $u = -5$ mm, $v = 3.5$ mm; and for $x = y = 0$, $u = v = 0$. Substitution of these conditions into Eqs. (a) yields the result

$$u = -0.002x - 0.003y, \qquad v = 0.001x + 0.0025y \tag{b}$$

With Eqs. (b), Eqs. (2-15.14) in Chapter 2 yield, for small-displacement theory,

$$\begin{aligned}
\epsilon_{11} &= \epsilon_x = \partial u/\partial x = -0.002\\
\epsilon_{22} &= \epsilon_y = \partial v/\partial y = 0.0025\\
\epsilon_{12} &= \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -0.001\\
\epsilon_{33} &= \epsilon_z = 0, \qquad \epsilon_{13} = \epsilon_{xz} = 0, \qquad \epsilon_{23} = \epsilon_{yz} = 0
\end{aligned} \tag{c}$$

With Eqs. (c) and Eqs. (2-9.1) in Chapter 2, strain components relative to any other set of axes (say, X, Y) may be computed. For example, let axes (X, Y) be obtained by a rotation in the (x, y) plane of 30° such that the direction cosines between axes (X, Y) and axes (x, y) are

$$a_{11} = \sqrt{3}/2, \qquad a_{12} = 1/2, \qquad a_{21} = -1/2, \qquad a_{22} = \sqrt{3}/2$$

(see Table 1-24.1 in Chapter 1). Then by Eqs. (c) and (2-9.1), we obtain the strain components E_X, E_Y, E_{XY} relative to axes (X, Y) as (noting that $E_Z = E_{XZ} = E_{YZ} = 0$)

$$\begin{aligned}
E_{11} &= E_X = \epsilon_{11}a_{11}^2 + \epsilon_{22}a_{12}^2 + 2\epsilon_{12}a_{11}a_{12} = -0.001741\\
E_{22} &= E_Y = \epsilon_{11}a_{21}^2 + \epsilon_{22}a_{22}^2 + 2\epsilon_{12}a_{21}a_{22} = 0.002241\\
E_{12} &= E_{XY} = \epsilon_{11}a_{11}a_{12} + \epsilon_{22}a_{12}a_{22} + \epsilon_{12}(a_{11}a_{22} + a_{12}a_{21}) = 0.001449
\end{aligned} \tag{d}$$

Note that $J_1 = \epsilon_x + \epsilon_y = E_X + E_Y = 0.0005$ and $J_2 = \epsilon_x\epsilon_y - \epsilon_{xy}^2 = E_X E_Y - E_{XY}^2 = -0.000006$.

Example 5-1.2. Stress–Strain–Strain Energy Density Relations: Plane Strain. The strain energy density for an anisotropic (crystalline) material subjected to a state of plane strain is given by

$$U = \tfrac{1}{2}(C_{11}\epsilon_{11}^2 + C_{22}\epsilon_{22}^2 + 2C_{33}\epsilon_{12}^2 + 2C_{12}\epsilon_{11}\epsilon_{22} + 4C_{13}\epsilon_{11}\epsilon_{12} + 4C_{23}\epsilon_{22}\epsilon_{12}) \quad \text{(a)}$$

We wish to determine the stress–strain relations for the material. By Eq. (4-4.21) in Chapter 4, $\sigma_{\alpha\beta} = \partial U/\partial\epsilon_{\alpha\beta}$, where U must be expressed symmetrically in terms of ϵ_{12}, ϵ_{21}. Thus, let

$$\epsilon_{12} = \tfrac{1}{2}(\epsilon_{12} + \epsilon_{21}) \quad \text{(b)}$$

Then substitution of Eq. (b) into Eq. (a) yields

$$\begin{aligned} U = \tfrac{1}{2}\Big[& C_{11}\epsilon_{11}^2 + C_{22}\epsilon_{22}^2 + 2C_{33}\left(\frac{\epsilon_{12}+\epsilon_{21}}{2}\right)^2 \\ & + 2C_{12}\epsilon_{11}\epsilon_{22} + 4C_{13}\epsilon_{11}\left(\frac{\epsilon_{12}+\epsilon_{21}}{2}\right) + 4C_{23}\epsilon_{22}\left(\frac{\epsilon_{12}+\epsilon_{21}}{2}\right)\Big] \end{aligned} \quad \text{(c)}$$

Then

$$\sigma_{11} = \frac{\partial U}{\partial \epsilon_{11}} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}(\epsilon_{12} + \epsilon_{21})$$

or, as $\epsilon_{12} = \epsilon_{21}$,

$$\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + 2C_{13}\epsilon_{12} \quad \text{(d)}$$

Similarly,

$$\begin{aligned} \sigma_{22} &= \frac{\partial U}{\partial \epsilon_{22}} = C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + 2C_{23}\epsilon_{12} \\ \sigma_{21} = \sigma_{12} &= \frac{\partial U}{\partial \epsilon_{12}} = C_{13}\epsilon_{11} + C_{23}\epsilon_{22} + C_{33}\epsilon_{12} \end{aligned} \quad \text{(e)}$$

Example 5-1.3. Integration of Plane Strain–Displacement Relations. The plane strain–displacement relations as given by Eqs. (5-1.4) are

$$\epsilon_x = \partial u/\partial x, \qquad \epsilon_y = \partial v/\partial y, \qquad \gamma_{xy} = \partial u/\partial y + \partial v/\partial x \quad \text{(a)}$$

where $u = u(x, y)$, $v = v(x, y)$. Elimination of (u, v) from Eqs. (a) yields the strain–compatibility relationship for plane strain

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{b}$$

In a plane strain problem, the strain components were determined as

$$\epsilon_x = Ax^2 + By^2, \qquad \epsilon_y = -Bx^2 - Ay^2, \qquad \gamma_{xy} = 0 \tag{c}$$

Substitution of Eqs. (c) into Eq. (b) shows that the strain components are compatible. By Eqs. (a) and (c),

$$\epsilon_x = \frac{\partial u}{\partial x} = Ax^2 + By^2, \qquad \epsilon_y = \frac{\partial v}{\partial y} = -Bx^2 - Ay^2, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{d}$$

Integration of the first two of Eqs. (d) yields

$$u = \tfrac{1}{3}Ax^3 + Bxy^2 + f_1(y), \qquad v = -Bx^2y - \tfrac{1}{3}Ay^3 + f_2(x) \tag{e}$$

where $f_1(y), f_2(x)$ are y and x functions of integration, respectively.

Substitution of Eqs. (e) into the third of Eqs. (d) yields $f_1'(y) + f_2'(x) = 0$, or

$$f_1'(y) = C, \qquad f_2'(x) = -C \tag{f}$$

where C is a constant. Hence, integration yields

$$f_1(y) = Cy + D, \qquad f_2(x) = -Cx + F \tag{g}$$

where C, D, and F are constants of integration that must be determined by specification of the rigid-body displacement (Section 2-15 in Chapter 2). Equations (e) and (g) yield the displacement components

$$\begin{aligned} u &= \tfrac{1}{3}Ax^3 + Bxy^2 + Cy + D \\ v &= -Bx^2y - \tfrac{1}{3}Ay^3 - Cx + F \end{aligned} \tag{h}$$

Problem Set 5-1

1. The two-dimensional body $OABC$ is held between two rigid frictionless walls as shown in Fig. P5-1.1. The region under the body is filled with a fluid at uniform pressure p. What are the boundary conditions required to solve for the stresses in body $OABC$? Neglect gravity.

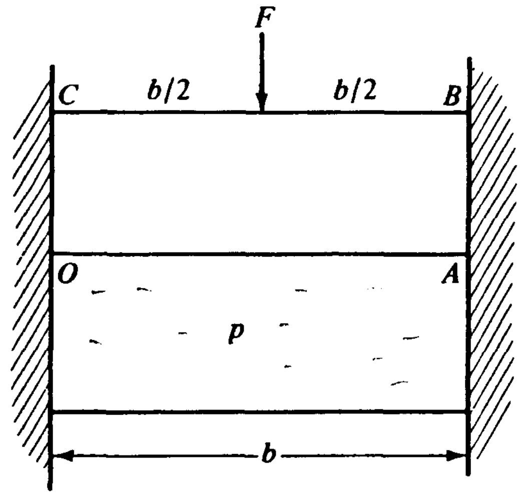


Figure P5-1.1

2. For a state of plane strain, $\sigma_x = f(y)$. Neglecting body forces, derive the most general equations for σ_x, σ_y, σ_z, and τ_{xy}.

3. The strain energy density U of a linearly elastic material is given by the relation $U = (\frac{1}{2}\lambda + G)J_1^2 - 2GJ_2$, where (λ, G) are the Lamé elastic constants and (J_1, J_2) are the first and second strain invariants.

   Employing the relationship between U and the stress components σ_x, derive the stress–strain relations for a state of plane strain relative to the (x, y) plane.

4. For a state of plane strain in an isotropic body, $\sigma_x = ay^2$, $\sigma_y = -ax^2$, $\tau_{xy} = 0$. The body forces and temperature are zero. Using small-displacement elasticity theory, compute the displacement components $u(x, y)$ and $v(x, y)$ (a is a constant). (See also Section 5-6.)

5. For a state of plane strain in an isotropic body,

$$\sigma_x = ay^2 + bx, \qquad \sigma_y = -ax^2 + by, \qquad \tau_{xy} = -b(x + y)$$

   The body forces and temperature are zero. Using small-displacement elasticity theory, compute the displacement components $u(x, y)$ and $v(x, y)$ (a and b constants). (See Section 5-6.)

6. Consider a rectangular region in the (x, y) plane subjected to a uniform stress σ in the x-direction along the edges parallel to the y axis. The (x, y) axes have origin at the center of the region.
   - **(a)** For an isotropic, homogeneous elastic material in this region, derive expressions for the (x, y) displacement components (u, v) in terms of (x, y) and arbitrary constants of integration by the theory of elasticity.
   - **(b)** Employ appropriate conditions at the origin ($x = y = 0$) to eliminate rigid-body displacements of the region and evaluate the arbitrary constants of integration.
   - **(c)** By elementary means of mechanics of materials, derive the displacement components and show that the results obtained in part (b) agree with these results.

5-2 Generalized Plane Stress

As described in Section 3-7 in Chapter 3, for certain kinds of loading, the equations of plane theory of elasticity apply to thin plates. We define a thin plate to be a prismatic member (for instance, a cylinder) of very small length or *thickness h*. The middle surface of the plate, located halfway between its ends and parallel to them, is taken as the (x, y) plane (see Fig. 5-2.1).

We assume that the faces (upper and lower ends) are free from external stresses and that the stresses that act on the edges of the plate are parallel to the faces and are distributed symmetrically with respect to the middle surface. Similar restrictions apply to the body forces. By symmetry, note that points that are originally in the middle surface of the plate lie in the middle surface after deformation. Also, because the plate is assumed thin, the displacement component w is small, and variations of the displacement components (u, v) through the thickness are small. Consequently, satisfactory results are obtained, if we treat the equilibrium problem of the plate in terms of mean values $\bar{u}$, $\bar{v}$, and $\bar{w}$ of displacement components (u, v, w) defined as follows:

$$\begin{aligned} \bar{u}(x, y) &= \frac{1}{h}\int_{-h/2}^{h/2} u(x, y, z)\, dz, \qquad \bar{v}(x, y) = \frac{1}{h}\int_{-h/2}^{h/2} v(x, y, z)\, dz \\ \bar{w}(x, y) &= \frac{1}{h}\int_{-h/2}^{h/2} w(x, y, z)\, dz \end{aligned} \tag{5-2.1}$$

where bars over letters denote mean values. In turn, substitution of Eqs. (5-2.1) into Eqs. (2-15.14) in Chapter 2 yields mean strains $\bar{\epsilon}_x$, $\bar{\epsilon}_y$, $\bar{\epsilon}_z$, $\bar{\gamma}_{xy}$, $\bar{\gamma}_{yz}$, $\bar{\gamma}_{xz}$.

Because it is assumed that $\tau_{xz} = \tau_{yz} = 0$ on the ends, that is, for $z = \pm h/2$ in the absence of body forces, it follows from the last of Eqs. (3-8.1) in Chapter 3 that for $z = \pm h/2$, $\partial\sigma_z/\partial z = 0$. This follows from the fact that because $\tau_{xz} = 0$ for $z = \pm h/2$, $\partial\tau_{xz}/\partial x = 0$ for $z = \pm h/2$, and because $\tau_{yz} = 0$ for $z = \pm h/2$, $\partial\tau_{yz}/\partial y = 0$ for $z = \pm h/2$.

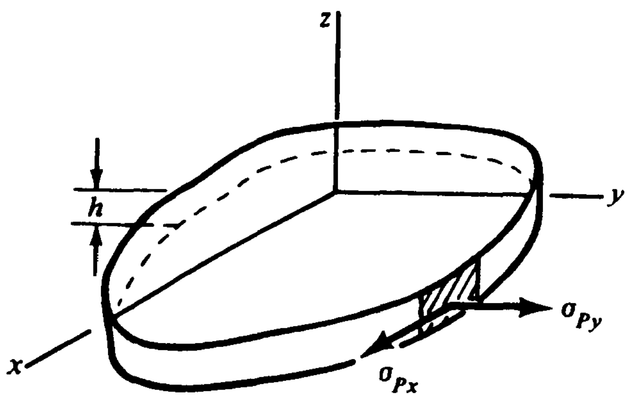


Figure 5-2.1

Hence, not only is σ_z zero for $z = \pm h/2$, but also its derivative with respect to z vanishes. Therefore, as the plate is thin, σ_z is small throughout the plate. These observations lead us naturally to the *approximation* that $\sigma_z = 0$ everywhere.

Analogously, we define mean values of stress components $(\sigma_x, \sigma_y, \tau_{xy})$ as follows:

$$\bar{\sigma}_x = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_x \, dz, \qquad \bar{\sigma}_y = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_y \, dz,$$
$$\bar{\tau}_{xy} = \frac{1}{h}\int_{-h/2}^{h/2} \tau_{xy} \, dz \tag{5-2.2}$$

Accordingly, the mean values of $(\sigma_x, \sigma_y, \tau_{xy})$ are independent of z.

Furthermore,

$$\frac{1}{h}\int_{-h/2}^{h/2} \frac{\partial}{\partial z}(\tau_{xz}) \, dz = \frac{1}{h}\tau_{xz}\Big|_{-h/2}^{h/2} = 0$$
$$\frac{1}{h}\int_{-h/2}^{h/2} \frac{\partial}{\partial z}(\tau_{yz}) \, dz = \frac{1}{h}\tau_{yz}\Big|_{-h/2}^{h/2} = 0 \tag{5-2.3}$$

Also, mean values of body forces are defined as

$$\bar{X} = \frac{1}{h}\int_{-h/2}^{h/2} X \, dz, \qquad \bar{Y} = \frac{1}{h}\int_{-h/2}^{h/2} Y \, dz, \qquad \bar{Z} = \frac{1}{h}\int_{-h/2}^{h/2} Z \, dz = 0 \tag{5-2.4}$$

Substitution of Eqs. (5-2.2), (5-2.3), and (5-2.4) into the first two of Eqs. (3-8.1) yields, after integration with respect to z (neglecting acceleration effects),

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{xy}}{\partial y} + \bar{X} = 0, \qquad \frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\sigma}_y}{\partial y} + \bar{Y} = 0 \tag{5-2.5}$$

From the stress–strain relations [Eqs. (4-6.5) in Chapter 4], it follows from $\sigma_z = \lambda e + 2G\epsilon_z = 0$ that (isotropic material)

$$\epsilon_z = -\frac{\lambda}{\lambda + 2G}(\epsilon_x + \epsilon_y) = -\frac{v}{1 - v}(\epsilon_x + \epsilon_y) \tag{5-2.6}$$

Substituting Eq. (5-2.6) into the first and second of Eqs. (4-6.5), we obtain

$$\sigma_x = \frac{2\lambda G}{\lambda + 2G}(\epsilon_x + \epsilon_y) + 2G\epsilon_x, \qquad \sigma_y = \frac{2\lambda G}{\lambda + 2G}(\epsilon_x + \epsilon_y) + 2G\epsilon_y \tag{5-2.7}$$

The fourth of Eqs. (4-6.5) is

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{5-2.8}$$

Taking the mean values of Eqs. (5-2.7) and (5-2.8), we obtain

$$\begin{aligned}\bar{\sigma}_x &= \bar{\lambda}(\bar{\epsilon}_x + \bar{\epsilon}_y) + 2G\bar{\epsilon}_x = \bar{\lambda}\bar{e} + 2G\bar{\epsilon}_x \\ \bar{\sigma}_y &= \bar{\lambda}(\bar{\epsilon}_x + \bar{\epsilon}_y) + 2G\bar{\epsilon}_y = \bar{\lambda}\bar{e} + 2G\bar{\epsilon}_y \\ \bar{\tau}_{xy} &= G\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right)\end{aligned} \tag{5-2.9}$$

where

$$\bar{\lambda} = \frac{2\lambda G}{\lambda + 2G} = \frac{\nu E}{1 - \nu^2}, \qquad \bar{\epsilon}_x = \frac{1}{h}\int_{-h/2}^{h/2} \epsilon_x \, dz, \qquad \bar{\epsilon}_y = \frac{1}{h}\int_{-h/2}^{h/2} \epsilon_y \, dz \tag{5-2.10}$$

Comparison of Eqs. (5-2.5) and (5-2.9) with Eqs. (5-1.6) shows that the mean values of displacement components (u, v) and the mean values of the stress components $(\sigma_x, \sigma_y, \tau_{xy})$ satisfy the same equations that govern the case of plane strain, the only difference being that λ is replaced by $\bar{\lambda}$ defined by Eq. (5-2.10). Additionally, the stress components σ_{nx}, σ_{ny} on the boundary of the plate are replaced by their mean values $\bar{\sigma}_{nx}$, $\bar{\sigma}_{ny}$ [see Eqs. (4-15.1) in Chapter 4].

Taking note of these facts, we may write equations of generalized plane stress without bars over the symbols. We keep in mind that components of stress, strain, and displacement are mean values and that λ is replaced by

$$\bar{\lambda} = \frac{2\lambda G}{\lambda + 2G} = \frac{\nu E}{1 - \nu^2}$$

Thus, we see that for plane strain and for generalized plane stress, we are led to the study of the following system of equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \tag{5-2.11}$$

$$\sigma_x = \lambda e + 2G\epsilon_x, \qquad \sigma_y = \lambda e + 2G\epsilon_y, \qquad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\gamma_{xy} \tag{5-2.12}$$

where

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{5-2.13}$$

Equations (5-2.11) may be written entirely in terms of strain components by substitution of Eqs. (5-2.12) into Eqs. (5-2.11). Equations (5-2.11) may also be written entirely in terms of displacement components by substitution of Eqs. (5-2.13) and (2-15.14) into Eqs. (5-2.12) and substitution of the result into Eqs. (5-2.11).

A more specialized state of stress, called *plane stress*, is obtained if we set $\sigma_z = \tau_{xz} = \tau_{yz} = Z = 0$ everywhere. Then the equilibrium equations are given by Eqs. (5-1.3).

Although in generalized plane stress, the mean values of the displacement components are independent of z, in a state of plane stress the displacement components (u, v, w) are not, in general, independent of z. In particular, we note that ϵ_z does not vanish and that it is defined by Eq. (5-2.6).

Furthermore, we observe that in a plate, a state of plane stress requires the body forces and the tractions at the edges to be distributed in certain special ways. It does not, however, require tractions on the faces of the plate.

Finally, we also remark that the average values of displacement in any problem of plane stress are the same as if the problem were one of generalized plane stress. Accordingly, the solution of problems of plane stress may be employed to examine effects produced by certain distributions of forces that do not produce plane-stress states, as any such problem can be solved by treating it as a plane-stress problem and by replacing λ by $\bar{\lambda}$ in the results. For example, this technique may be employed in problems of equilibrium of a thin plate deformed by forces applied in the plane of the plate. Although the actual values of stress and displacement are not determined by this procedure (unless the forces actually produce a state of plane stress), the average values across the thickness of the plate are obtained. Moreover, average values are the usual quantities measured experimentally.

Example 5-2.1. Plane Stress. Consider a plane stress problem relative to the (x, y) plane, that is,

$$\begin{aligned}&\sigma_x = \sigma_x(x, y), \qquad \sigma_y = \sigma_y(x, y), \qquad \tau_{xy} = \tau_{xy}(x, y)\\&\sigma_z = \tau_{xz} = \tau_{yz} = 0\end{aligned} \tag{a}$$

The corresponding strain components [Eqs. (4-6.8) or (5-3.6)] for constant temperature T are

$$\begin{aligned}&\epsilon_{11} = \epsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y), \qquad \epsilon_{22} = \frac{1}{E}(\sigma_y - v\sigma_x)\\&2\epsilon_{12} = \gamma_{xy} = \frac{2(1+v)}{E}\sigma_{12} = \frac{2(1+v)}{E}\tau_{xy}\\&\epsilon_{33} = \epsilon_z = -\frac{v}{E}(\sigma_x + \sigma_y)\end{aligned} \tag{b}$$

For a particular plane stress problem, it has been found that the (x, y) displacement components (u, v) are given by the equations

$$\begin{aligned}u &= a_1 + a_2x + a_3y + a_4xy\\v &= b_1 + b_2x + b_3y + b_4xy\end{aligned} \tag{c}$$

where a_i, b_i are constants. We wish to determine the corresponding small-displacement nonzero strain components [Eqs. (b)] and the stress components [Eqs. (a)] as functions of (x, y).

By Eqs. (2-15.4) and (c),

$$\begin{aligned} \epsilon_{11} &= \epsilon_x = \frac{\partial u}{\partial x} = a_2 + a_4 y, \qquad \epsilon_{22} = \epsilon_y = \frac{\partial v}{\partial y} = b_3 + b_4 x \\ 2\epsilon_{12} &= \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + b_2 + a_4 x + b_4 y \end{aligned} \tag{d}$$

To determine ϵ_z we need σ_x, σ_y. Hence, substitution of Eqs. (d) into Eqs. (b) and solution for (σ_x, σ_y) yields

$$\begin{aligned} \sigma_x &= \frac{E}{1 - \nu^2}(a_2 + \nu b_3 + \nu b_4 x + a_4 y) \\ \sigma_y &= \frac{E}{1 - \nu^2}(\nu a_2 + b_3 + b_4 x + \nu a_4 y) \end{aligned} \tag{e}$$

Then, by the last two of Eqs. (b) and Eqs. (d) and (e),

$$\begin{aligned} \tau_{xy} &= \frac{E}{2(1 + \nu)}(a_3 + b_2 + a_4 x + b_4 y) \\ \epsilon_z &= -\frac{\nu}{1 - \nu}(a_2 + b_3 + b_4 x + a_4 y) \end{aligned} \tag{f}$$

Equations (d), (e), and (f) determine the nonzero stress and strain components.

Problem Set 5-2

1. Repeat Problem 5-13 for the case of plane stress.
2. A material is isotropic and elastic. Body forces and temperature are zero. All stress components are zero except τ_{xy}. Using small-displacement theory, determine the most general form for τ_{xy}.
3. Consider a plane stress problem relative to the (x, y) plane. At a point P in the (x, y) plane the normal stresses on three planes perpendicular to the (x, y) plane and forming angles 120° relative to each other are $4C$, $3C$, and $2C$, respectively, in the counterclockwise direction, with the direction of the stress $4C$ coincident with the positive x axis. Determine the principal stresses at P.
4. The following stress array is proposed as a solution to a certain *equilibrium* problem of a plane body bounded in the region $-L/2 \le x \le L/2$, $-h/2 \le y \le h/2$:

$$\begin{aligned} \sigma_x &= Ay + Bx^2 y + Cy^3, \qquad \sigma_y = Dy^3 + Ey + F, \\ \tau_{xy} &= (G + Hy^2)x, \qquad \sigma_z = \tau_{xz} = \tau_{yz} = 0 \end{aligned}$$

where (x, y, z) are rectangular Cartesian coordinates and $A, B, \ldots, H$ are nonzero constants. Determine the conditions under which this array is a possible equilibrium solution.

It is proposed that the region be loaded such that $\tau_{xy} = 0$ for $y = \pm h/2$, $\sigma_y = 0$ for $y = h/2$, $\sigma_y = -\sigma$ (σ = constant) for $y = -h/2$, and $\sigma_x = 0$ for $x = \pm L/2$. Determine whether the proposed stress array may satisfy these conditions.

5. A flat plate is in a state of biaxial tension. The principal stresses are σ_x and σ_y (see Fig. P5-2.5). Two electrical strain gages are located as shown. The angle α is given by

$$\cos\alpha = \sqrt{\frac{1}{1+\nu}} \qquad \sin\alpha = \sqrt{\frac{\nu}{1+\nu}}$$

Assume that the material is linearly elastic and isotropic. Prove that the principal stresses may be read directly (except for a constant factor) as the strains in the direction of the two strain gages 1 and 2.

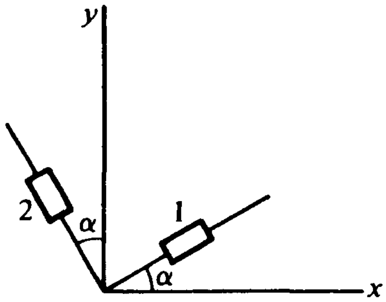


Figure P5-2.5

6. A semi-infinite space is subjected to a uniformly distributed pressure over its entire bounding plane (Fig. P5-2.6). Consider an infinitesimal volume element $ABCD$ at some

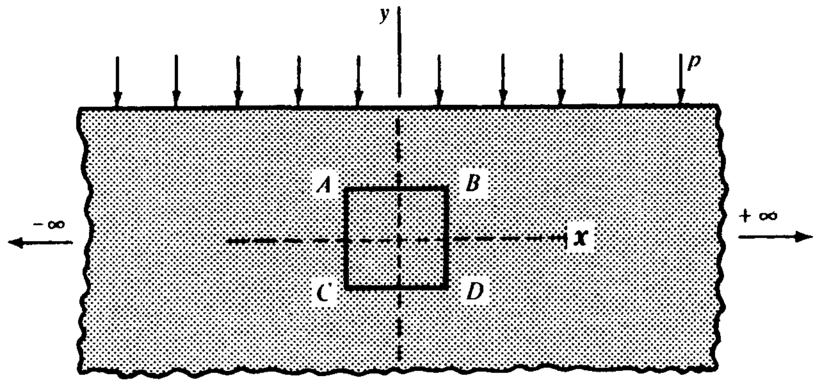


Figure P5-2.6

distance from the bounding plane. The normal stress on surface AB is $\sigma_y = \sigma$. In terms of the appropriate material properties and σ, derive expressions for the normal stress components σ_x, σ_z that act on the volume element (axis z is perpendicular to the x, y plane). *Hint*: What are the values of the strain components ϵ_x, ϵ_z?

5-3 Compatibility Equation in Terms of Stress Components

Equations (5-2.11) and one supplementary condition (the compatibility condition), which ensures that there exist two displacement components (u, v) related to the three stress components $(\sigma_x, \sigma_y, \tau_{xy})$ through Eqs. (5-2.12), comprise the equations of plane elasticity. The compatibility equation may be derived from Eqs. (4-14.2) or from Eqs. (2-16.1).

Plane Strain. Consider the state of plane strain. Such a state is defined by the conditions that ϵ_x, ϵ_y, γ_{xy} are independent of z, and $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$. Hence, the compatibility conditions reduce to the single equation [Eq. (2-16.1) in Chapter 2]

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{5-3.1}$$

Also, Eqs. (4-6.8) with Eqs. (5-1.5) become

$$\begin{aligned} \epsilon_x &= \frac{1}{2G}\left[\sigma_x - \frac{\lambda}{2(\lambda + G)}(\sigma_x + \sigma_y)\right] \\ \epsilon_y &= \frac{1}{2G}\left[\sigma_y - \frac{\lambda}{2(\lambda + G)}(\sigma_x + \sigma_y)\right] \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy} \end{aligned} \tag{5-3.2}$$

Substitution of Eqs. (5-3.2) into Eq. (5-3.1) yields

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu \nabla^2(\sigma_x + \sigma_y) = 2\,\frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} \tag{a}$$

Equations (5-1.3) yield

$$-2\,\frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} = \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \tag{b}$$

Substitution of Eq. (b) into Eq. (a) yields after simplification

$$\begin{aligned}\nabla^2(\sigma_x+\sigma_y) &= -\frac{2(\lambda+G)}{\lambda+2G}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\\ &= -\frac{1}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\end{aligned} \tag{5-3.3}$$

Equations (5-2.11) and (5-3.3) represent the equations of plane strain. The equations of generalized plane stress are obtained from these equations if mean values of stress and body force are used and if λ is replaced by

$$\bar{\lambda}=\frac{2\lambda G}{\lambda+2G}=\frac{\nu E}{1-\nu^2}$$

Generalized Plane Stress. For generalized plane stress, $\sigma_z = 0$. Hence, by the third of Eqs. (4-6.5) in Chapter 4,

$$\epsilon_z=-\frac{\nu}{1-\nu}(\epsilon_x+\epsilon_y)=-\frac{\nu}{1-\nu}(u_x+v_y) \tag{5-3.4}$$

By Eqs. (5-3.4), we may eliminate ϵ_z from the first two of Eqs. (4-6.5). Then, on taking mean values, we obtain

$$\begin{aligned}&\bar{\sigma}_x=\bar{\lambda}\bar{e}+2G\bar{\epsilon}_x, \qquad \bar{\sigma}_y=\bar{\lambda}\bar{e}+2G\bar{\epsilon}_y, \qquad \bar{\tau}_{xy}=G\bar{\gamma}_{xy}\\ &\bar{e}=\bar{\epsilon}_x+\bar{\epsilon}_y, \qquad \bar{\lambda}=\frac{2G\nu}{1-\nu}\end{aligned} \tag{5-3.5}$$

or, alternatively, in terms of E, ν,

$$\begin{aligned}\bar{\sigma}_x&=\frac{E}{1-\nu^2}(\bar{\epsilon}_x+\nu\bar{\epsilon}_y)\\ \bar{\sigma}_y&=\frac{E}{1-\nu^2}(\bar{\epsilon}_y+\nu\bar{\epsilon}_x)\\ \bar{\tau}_{xy}&=\frac{E}{2(1+\nu)}\bar{\gamma}_{xy}\end{aligned} \tag{5-3.5a}$$

The inverse relations are

$$E\bar{\epsilon}_x=\bar{\sigma}_x-\nu\bar{\sigma}_y, \qquad E\bar{\epsilon}_y=\bar{\sigma}_y-\nu\bar{\sigma}_x, \qquad G\bar{\gamma}_{xy}=\bar{\tau}_{xy} \tag{5-3.6}$$

The mean strain components evidently satisfy the compatibility conditions [Eqs. (2-16.1) or (5-3.1)]. With Eq. (5-3.6), Eq. (5-3.1) may be expressed in terms of stress as

$$\nabla^2(\bar{\sigma}_x + \bar{\sigma}_y) = -(1+\nu)\left(\frac{\partial \bar{X}}{\partial x} + \frac{\partial \bar{Y}}{\partial y}\right) \tag{5-3.7}$$

In view of the principle of superposition, body forces can be eliminated from consideration if a particular solution is found. We must then solve a problem with no body forces but with altered boundary conditions. For constant body forces or centrifugal body forces, particular solutions are easily found. Consequently, let us consider cases in which body forces are absent. Then the compatibility conditions for generalized plane stress and strain [Eqs. (5-3.7) and (5-3.3)] are identical. The stress–strain relations are the same in both cases, except that $\bar{\lambda}$ replaces λ in problems of generalized plane stress.

In terms of the Airy stress function F (see Section 5-4), the problem, in either case, reduces to the solution $\nabla^2\nabla^2 F = 0$ in the absence of body forces. Furthermore, by the principle of superposition, any solution of an axial stress problem may be superimposed on a plane-strain solution. For the general plane orthogonal curvilinear coordinate system the defining equation for F is obtained by specializing the expression for ∇^2 for the plane, that is, setting $h_3 = 1$ and $\partial/\partial w = 0$ [see Section 1-22 and Eq. (1-22.13) in Chapter 1].

Summary of Equations of Plane Elasticity. For convenience we summarize the equations of the plane theory of elasticity for an isotropic, homogeneous material. Also, for completeness we include the effects of body force (X, Y) and temperature T. All quantities are considered to be functions of (x, y) coordinates.

Plane Strain. The stress–strain–temperature relations are

$$\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_x + \nu\epsilon_y - (1+\nu)kT] \\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_x + (1-\nu)\epsilon_y - (1+\nu)kT] \\
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \\
\sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}[\nu(\epsilon_x + \epsilon_y) - (1+\nu)kT] \\
e &= u_x + v_y = \epsilon_x + \epsilon_y \\
\epsilon_z &= \gamma_{xz} = \gamma_{yz} = \tau_{xz} = \tau_{yz} = 0
\end{aligned} \tag{5-3.8}$$

The compatibility relation in terms of stress components is

$$\nabla^2(\sigma_x + \sigma_y) + \frac{E}{1-v}\nabla^2(kT) + \frac{1}{1-v}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0 \tag{5-3.9}$$

where E (Young's modulus) and v (Poisson's ratio) are constants and where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Plane Stress. The stress–strain–temperature relations are

$$\begin{aligned}
\sigma_x &= \frac{E}{1-v^2}[\epsilon_x + v\epsilon_y - (1+v)kT] \\
\sigma_y &= \frac{E}{1-v^2}[v\epsilon_x + \epsilon_y - (1+v)kT] \\
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+v)}\gamma_{xy} \\
\epsilon_z &= -\frac{1}{1-v}[v(\epsilon_x + \epsilon_y) - (1+v)kT] \\
e &= \epsilon_x + \epsilon_y + \epsilon_z = \frac{1}{1-v}[(1-2v)(\epsilon_x + \epsilon_y) + (1+v)kT] \\
\sigma_z &= \tau_{xz} = \tau_{yz} = \gamma_{xz} = \gamma_{yz} = 0
\end{aligned} \tag{5-3.10}$$

The compatibility relation in terms of stress components is

$$\nabla^2(\sigma_x + \sigma_y) + E\nabla^2(kT) + (1+v)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0 \tag{5-3.11}$$

Equations (5-3.9) and (5-3.11), subject to appropriate boundary conditions, constitute the equations from which the sum of stress components σ_x, σ_y is determined. Mathematically speaking, Eqs. (5-3.9) and (5-3.11) are equivalent, as we may write

$$\nabla^2(\sigma_x + \sigma_y) + K_1\nabla^2(kT) + K_2\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0 \tag{5-3.12}$$

where for plane strain $K_1 = E/(1-v)$, $K_2 = 1/(1-v)$, and for plane stress $K_1 = E$, $K_2 = 1 + v$. In other words, Eq. (5-3.9) is obtained from Eq. (5-3.11) by the substitutions

$$E \to \frac{E}{1-v}, \qquad 1 + v \to \frac{1}{1-v} \tag{5-3.13}$$

Accordingly, the mathematical problems of plane strain and plane stress are equivalent.

Example 5-3.1. Compatibility Conditions for Plane Problems. Relation to Three-Dimensional Compatibility Relations. A Caution. In Section 5-3 we noted that the strain compatibility equation may be represented in terms of stress components. In particular, in the absence of body forces and temperature, the compatibility relation for plane problems reduces to [Eqs. (5-3.9) and (5-3.12)]

$$\nabla^2(\sigma_x + \sigma_y) = 0 \tag{a}$$

whereas for the three-dimensional problem, the compatibility relations in terms of stress components are given by Eqs. (4-14.2) in Chapter 4. It is possible that a two-dimensional state of stress may satisfy Eq. (a) but may not satisfy all of Eqs. (4-14.2). For example, a two-dimensional solution for a cantilever beam [see Eq. (b), Example 5-7.1] is given by the stress state

$$\begin{aligned} \sigma_x &= A - 2Bxy, \qquad \sigma_y = 0 \\ \tau_{xy} &= -B(c^2 - y^2) \end{aligned} \tag{b}$$

where A, B, and c are constants. Equation (a) is satisfied by Eqs. (b). However, if Eqs. (b) are substituted in Eqs. (4-14.2), it is found that all the equations are satisfied identically except for the equation

$$\nabla^2\tau_{xy} + \frac{1}{1+v}\frac{\partial^2 I_1}{\partial x\,\partial y} = 0 \tag{c}$$

Equations (b) and (c) yield the result

$$1 + v = 1 \tag{d}$$

Thus, Eq. (d) cannot be satisfied unless Poisson's ratio $v = 0$, which is not possible for known materials. Hence, a solution may be compatible in the two-dimensional state but not in the three-dimensional state.

Problem Set 5-3

1. Consider a wedge hanging vertically in a gravity field of acceleration g (Fig. P5-3.1). The following elasticity solution for the stress problem of the wedge is proposed: $\sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = 0$, $\sigma_z = \frac{1}{2}\rho gz$, $\tau_{xz} = \frac{1}{2}\rho gx$. Discuss this proposed solution.

2. Consider a beam in the region $-h/2 \le y \le h/2$, $-b/2 \le z \le b/2$, $0 \le x \le L$. Assume plane stress in the (x, y) plane, with zero body forces. The stress component normal to the plane perpendicular to the x axis is $\sigma_x = -My/I$, where $M = M(x)$ is a function of x only,

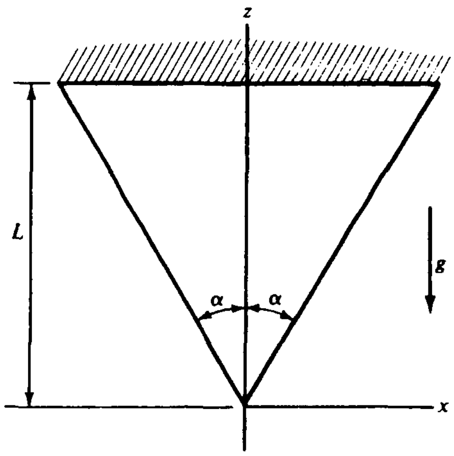


Figure P5-3.1

and $I = bh^3/12$. Derive expressions for σ_y and τ_{xy} subject to the boundary conditions $\tau_{xy} = 0$ for $y = \pm h/2$ and $\sigma_y = 0$ for $y = h/2$. What restriction, if any, must be placed on M in order that the derived state of stress be compatible? What can be said about σ_y at $y = -h/2$?

3. Given the following stress state:

$$\sigma_x = C[y^2 + \nu(x^2 - y^2)], \qquad \tau_{xy} = -2C\nu xy$$
$$\sigma_y = C[x^2 + \nu(y^2 - x^2)], \qquad \tau_{yz} = \tau_{xz} = 0$$
$$\sigma_z = C\nu(x^2 + y^2)$$

Discuss the possible reasons for which this stress state may not be a solution of a problem in elasticity.

5-4 Airy Stress Function

Simply Connected Regions. For the plane theory of elasticity, the equilibrium equations [Eqs. (3-8.1) in Chapter 3] reduce to two equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \qquad (5\text{-}4.1)$$

As noted in Section 5-3, we may initially ignore body forces (X, Y) and seek solutions to Eqs. (5-4.1) modified accordingly. Then the effects of body forces may be superimposed. However, in the case of body forces derivable from a potential function $V(\nabla^2 V = 0)$, such that

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y} \tag{5-4.2}$$

we may incorporate the effects of body force directly. Thus, Eqs. (5-4.1) and (5-4.2) yield

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \sigma_y'}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \tag{5-4.3}$$

where

$$\sigma_x' = \sigma_x - V, \qquad \sigma_y' = \sigma_y - V \tag{5-4.4}$$

Now, for simply connected regions, we note that the first of Eqs. (5-4.3) represents the necessary and sufficient condition that there exist a function $\phi(x, y)$ such that (see Section 1-19 in Chapter 1)

$$\frac{\partial \phi}{\partial y} = \sigma_x', \qquad \frac{\partial \phi}{\partial x} = -\tau_{xy} \tag{5-4.5}$$

The second of Eqs. (5-4.3) represents the necessary and sufficient condition that there exist a function $\theta(x, y)$ such that

$$\frac{\partial \theta}{\partial x} = \sigma_y', \qquad \frac{\partial \theta}{\partial y} = -\tau_{xy} \tag{5-4.6}$$

Comparison of the two expressions for τ_{xy} shows that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \theta}{\partial y} \tag{5-4.7}$$

In turn, Eq. (5-4.7) is the necessary and sufficient condition that there exist a function $F(x, y)$ such that

$$\phi = \frac{\partial F}{\partial y}, \qquad \theta = \frac{\partial F}{\partial x} \tag{5-4.8}$$

Substitution of Eq. (5-4.8) into Eqs. (5-4.5) and (5-4.6) shows that there always exists a function F such that for body forces represented by Eqs. (5-4.2), stress components in the plane theory of elasticity may be expressed in the form

$$\sigma_x' = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y' = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y}$$

Alternatively, by Eqs. (5-4.4) we have

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + V, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} + V, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y} \tag{5-4.9}$$

The function F is called the *Airy stress function* in honor of G. B. Airy, who first noted this relation.

Because it was assumed that the stresses σ_x, σ_y, τ_{xy} are single-valued and continuous together with their second-order derivatives [note the compatibility equations in terms of stress components Eq. (5-3.3)], the function F must possess continuous derivatives up to and including fourth order. These derivatives, from the second order on up, must be single-valued functions throughout the region occupied by the body [see Eqs. (5-4.9)].

Conversely, if F has these properties, the functions σ_x, σ_y, τ_{xy} defined in terms of F by Eqs. (5-4.9) will satisfy Eq. (5-4.1), provided body forces are defined by Eqs. (5-4.2). Additionally, to ensure that the stresses so determined correspond to an actual deformation, the compatibility conditions for the plane theory of elasticity must be satisfied. For body forces defined by Eq. (5-4.2) (or for constant body forces), this condition becomes [see Eq. (5-3.3) or (5-3.7)]

$$\nabla^2(\sigma_x + \sigma_y) = 0 \tag{5-4.10}$$

Adding the first two of Eqs. (5-4.9), we note that

$$\sigma_x + \sigma_y = \nabla^2 F + 2V \tag{5-4.11}$$

Substitution of Eq. (5-4.11) into Eq. (5-4.10) yields (because $\nabla^2 V = 0$)

$$\nabla^2\nabla^2 F = \frac{\partial^4 F}{\partial x^4} + 2\,\frac{\partial^4 F}{\partial x^2\, \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{5-4.12}$$

Equation (5-4.12) is the compatibility condition of the plane theory of elasticity with constant body forces or body forces derivable from a potential function [Eq. (5-4.2)] in terms of the stress function F.

Equations of the form of Eq. (5-4.12) are called *biharmonic*. Solutions of Eq. (5-4.12) are called biharmonic functions (Churchill et al., 1989). Some well-known solutions to Eq. (5-4.12) are, in rectangular coordinates,

$$\begin{gathered} y,\quad y^2,\quad y^3,\quad x,\quad x^2,\quad x^3,\quad xy,\quad x^2y,\quad xy^2,\quad x^3y,\quad xy^3, \\ x^2-y^2,\qquad x^4-y^4,\qquad x^2y^2-\tfrac{1}{3}y^4,\ldots \\ \cos\lambda y\ \cosh\lambda x,\qquad \cosh\lambda y\ \cos\lambda x,\qquad y\cos\lambda y\ \cosh\lambda x \\ y\cosh\lambda y\ \cos\lambda x,\qquad x\cos\lambda y\ \cosh\lambda x,\qquad x\cosh\lambda y\ \cos\lambda x \end{gathered} \tag{5-4.13}$$

By the above analysis, the problem of plane elasticity has been reduced to seeking solutions to Eq. (5-4.12) such that the stress components [Eqs. (5-4.9)] satisfy the boundary conditions. A number of problems may be solved by using simple linear combinations of polynomials in x and y (see Section 5-7).

Airy Stress Function with Body Forces and Temperature Effects. More generally, Eq. (5-4.12) may be written to include temperature effects [see Eqs. (5-3.9), (5-3.11), and (5-3.12)]. Body forces derivable from a potential function [Eq. (5-4.2)] do not affect Eq. (5-4.12). Hence, for potential body forces, Eq. (5-4.12) generalized to include temperature effects is [see Eq. (5-3.12)]

$$\nabla^2\nabla^2 F + C\nabla^2(kT) = 0 \tag{5-4.12a}$$

where $C = E$ for plane stress and $C = E/(1-\nu)$ for plane strain. Cases of more general body forces ordinarily must be treated individually.

Problem. Verify that the functions listed in Eq. (5-4.13) satisfy Eq. (5-4.12).

Boundary Conditions. It is frequently convenient to have the stress boundary conditions [Eqs. (4-15.1) in Chapter 4] expressed in terms of the Airy stress function. For simply connected regions, Eqs. (4-15.1) may be transformed as follows.

Consider region $G: (x, y)$ bounded by the curve $\mathbf{\Gamma}$ (Fig. 5-4.1). The unit normal vector (+outward) is

$$\mathbf{n} = (l, m, n) = \left(\frac{dy}{ds},\ -\frac{dx}{ds},\ 0\right) \tag{5-4.14}$$

where s denotes arc length measured from some arbitrary point P on $\mathbf{\Gamma}$. The unit tangent vector to $\mathbf{\Gamma}$ is denoted by $\mathbf{t}$, the positive direction of $\mathbf{t}$ being such that $(\mathbf{n}, \mathbf{t})$ form a right-handed system.

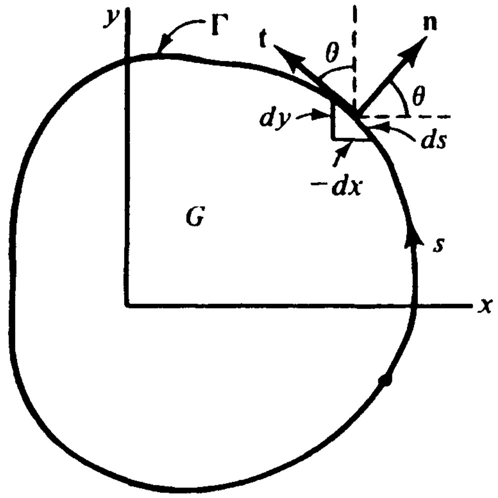


Figure 5-4.1

For the plane theory of elasticity with respect to the (x, y) plane, the boundary conditions [Eqs. (4-15.1)] reduce to

$$\sigma_{nx} = l\sigma_x + m\tau_{xy}, \qquad \sigma_{ny} = l\tau_{xy} + n\sigma_y \tag{5-4.15}$$

Substitution of Eqs. (5-4.14) into Eqs. (5-14.5) yields

$$\sigma_{nx} = \sigma_x \frac{dy}{ds} - \tau_{xy}\frac{dx}{ds}, \qquad \sigma_{ny} = \tau_{xy}\frac{dy}{ds} - \sigma_y \frac{dx}{ds} \tag{5-4.16}$$

By Eqs. (5-4.16), (5-4.5), and (5-4.6), we eliminate σ_x, σ_y, τ_{xy} to obtain

$$\sigma_{nx} = \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{\partial \phi}{ds}$$

$$\sigma_{ny} = -\frac{\partial \theta}{\partial y}\frac{dy}{ds} - \frac{\partial \theta}{\partial x}\frac{dx}{ds} = -\frac{d\theta}{ds}$$

or, multiplying by ds, we get

$$\begin{aligned} \sigma_{nx}\, ds &= \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = d\phi \\ -\sigma_{ny}\, ds &= \frac{\partial \theta}{\partial x}\, dx + \frac{\partial \theta}{\partial y}\, dy = d\theta \end{aligned} \tag{5-4.17}$$

Integration of Eq. (5-4.17) yields [with Eq. (5-4.8)]

$$\begin{aligned}\phi &= \frac{\partial F}{\partial y} = \int \sigma_{nx}\, ds = \int_0^l \sigma_{nx}\, ds + C_1 = R_x + C_1 \\ \theta &= \frac{\partial F}{\partial x} = -\int \sigma_{ny}\, ds = -\int_0^l \sigma_{ny}\, ds + C_2 = -R_y + C_2\end{aligned} \tag{5-4.18}$$

where (R_x, R_y) denote the (x, y) projections of the total force acting on Γ from 0 to l, and (C_1, C_2) are constants. Equations (5-4.18) express the stress boundary conditions [Eqs. (4-15.1)] in terms of derivatives of the Airy stress function F.

The stress boundary conditions may be interpreted physically in terms of the net force and net moment at $s = l$ resulting from the stress distributed on the boundary from $s = 0$ to $s = l$. For example, recall that by definition the total differential dF of F is

$$dF = \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy \tag{5-4.19}$$

Substitution of Eqs. (5-4.18) into Eq. (5-4.19) yields, after integration,

$$F(l) = \int_0^l dF = \int_0^l (-R_y\, dx + R_x\, dy) + C_1(y - y_0) + C_2(x - x_0) + C_3$$

Because linear terms in F do not contribute to the stress components [Eq. (5-4.9) with $V = 0$], we take $C_1 = C_2 = C_3 = 0$. Then integration by parts yields [with Eqs. (5-4.18)]

$$\begin{aligned}F(l) &= \int_0^l (-R_y\, dx + R_x\, dy) \\ &= (-xR_y + yR_x)\Big|_0^l - \int_0^l (-x\, dR_y + y\, dR_x) \\ &= -x_1 R_y(l) + y_1 R_x(l) + \int_0^l (x\sigma_{ny} - y\sigma_{nx})\, ds \\ &= -\int_0^l (x_l - x)\sigma_{ny}\, ds + \int_0^l (y_l - y)\sigma_{nx}\, ds = M_l\end{aligned} \tag{5-4.20}$$

where M_l denotes the moment with respect to $P{:}\,(s = l)$ of boundary forces on Γ from the point $P{:}\,(s = 0)$ to the point $P{:}\,(s = l)$. Thus, Eq. (5-4.20) shows that the value $F(l)$ of the Airy stress function at $s = l$ relative to its value at $s = 0$, is equal to the net moment of the boundary forces on Γ from the point $s = 0$ to the point $s = l$.

Equation (5-4.20) replaces one of the boundary conditions [Eqs. (5-4.18)]. To obtain a second equation, consider the directional derivative of the Airy stress function in the direction of **n** (Fig. 5-4.1). We have [see Section 1-8 in Chapter 1 and Eqs. (5-4.14) and (5-4.18)]

$$\begin{aligned}\frac{dF(l)}{dn} &= \mathbf{n} \cdot \text{grad}\, F \\ &= \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) \cdot (-R_y(l), R_x(l)) \\ &= -\left(\frac{dx}{ds}, \frac{dy}{ds}\right) \cdot \left[\int_0^l \sigma_{nx}\, ds, \int_0^l \sigma_{ny}\, ds\right] \\ &= -\mathbf{t} \cdot \mathbf{R} \end{aligned} \tag{5-4.21}$$

where **R** denotes the resultant external force acting on $\boldsymbol{\Gamma}$ from the point $s = 0$ to the point $s = l$. Hence, the normal derivative of F at point $s = l$ is equal to the negative of the projection **R** on the tangent **t** to the curve $\boldsymbol{\Gamma}$ at point $s = l$.

Equations (5-4.20) and (5-4.21) serve as boundary conditions in terms of the Airy stress function F. If the boundary $\boldsymbol{\Gamma}$ is free of external forces, Eqs. (5-4.20) and (5-4.21) yield

$$F(l) = 0, \qquad \frac{dF(l)}{dn} = 0 \tag{5-4.22}$$

Multiply Connected Regions. The above argument assumes that derivatives $G = G(x, y)$ of second order or higher of the Airy stress function are single-valued functions of (x, y). Hence, it is restricted to simply connected regions for which

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad P = \frac{\partial G}{\partial x}, \qquad Q = \frac{\partial G}{\partial y} \tag{5-4.23}$$

are necessary and sufficient conditions for the existence of G (see Section 1-19 in Chapter 1). For multiply connected regions with bounding contours L_k (Fig. 5-4.2), the condition (5-4.23) is only a necessary condition for the existence of the single-valued functions $G(x, y)$. For a multiply connected region, in addition to Eq. (5-4.23), the conditions

$$J_k = \int_{L_k} P\, dx + Q\, dy = 0, \qquad k = 1, 2, 3, \ldots, m \tag{5-4.24}$$

are also required.

Accordingly, in order that the derivatives $G(x, y)$ of second order or higher of the Airy stress function $F(x, y)$ be single-valued, it is necessary and sufficient that in addition to Eq. (5-4.23), the following conditions (Muskhelishvili, 1975) hold:

$$J_1 = J_2 = \cdots = J_k = \cdots = J_m = 0 \tag{5-4.25}$$

where J_k is defined by Eq. (5-4.24).

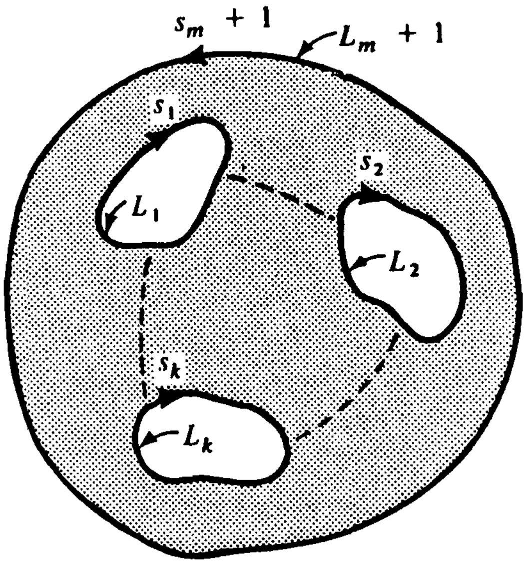


Figure 5-4.2

The defining equations for the Airy stress function [Eqs. (5-4.12), (5-4.20), (5-4.21), (5-4.25)] may be expressed for the general plane orthogonal curvilinear coordinate system by specializing the results of Section 1-22 in Chapter 1 for the plane.

Equations (5-4.23) and (5-4.25) ensure the single-valuedness of the stress components σ_x, σ_y, τ_{xy}. However, they do not assure the existence of single-valued displacement components (u, v), as these components are obtained by an integration of stress (or strain) quantities, this integration process possibly yielding multivalued terms. Accordingly, if we require single-valued displacement, we must select the arbitrary functions (or constants) that result in the expressions for (u, v) in such a fashion that the single-valuedness of displacement is ensured. Although we ordinarily require that the displacement be single valued, the concept of multivalued displacement components may be interpreted in a physical sense and finds an application through Volterra's theory of dislocation (see Love, 1944, pp. 221–228).

Example 5-4.1. Plane Theory of Thermoelasticity. Concept of Displacement Potential. In the absence of body forces, the plane theory of thermoelasticity may be reduced to the problem of determining a stress function F such that [Eq. (5-4.12a)]

$$\nabla^2\nabla^2 F = -C\nabla^2(kT) \tag{E5-4.1}$$

where $C = E$ for the plane-stress state and $C = E/(1 - v)$ for the plane strain state. In addition to Eq. (E5-4.1), the stress function F must satisfy appropriate boundary conditions (see Section 5-4). In general, the solution of Eq. (E5-4.1) subject to specific boundary conditions is a difficult mathematical problem, although in certain special cases simple solutions may be obtained. A general solution of Eq. (E5-4.1) may be obtained by adding a particular solution, for which the right-hand side of Eq. (E5-4.1) is satisfied identically, to the solution (complementary solution) of $\nabla^2\nabla^2 F = 0$. A method of obtaining a particular integral of Eq. (E5-4.1) has been outlined by Goodier (1937). The method is frequently referred to as the method of displacement potential, because displacement representations and certain concepts from potential theory are employed.

Following Goodier, we represent the plane theory of thermoelasticity in terms of displacement components. Initially, we consider the case of plane stress, the results for plane strain being obtained by a simple transformation of material constants.

Let (x, y) denote rectangular Cartesian coordinates. Let (u, v) denote displacement components in the (x, y) directions, respectively. In terms of (u, v), the stress components for plane stress are (see Section 4-12 in Chapter 4)

$$
\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\,\frac{\partial v}{\partial y} - (1+\nu)kT\right] \\
\sigma_y &= \frac{E}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\,\frac{\partial u}{\partial x} - (1+\nu)kT\right] \\
\tau_{xz} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}
\tag{E5-4.2}
$$

Substitution of Eqs. (E5-4.2) into the equilibrium equations for plane stress yields, in the absence of body force [see Eq. (5-4.1)],

$$
\begin{aligned}
\frac{\partial e}{\partial x} + \frac{1-\nu}{1+\nu}\nabla^2 u &= 2k\,\frac{\partial T}{\partial x} \\
\frac{\partial e}{\partial y} + \frac{1-\nu}{1+\nu}\nabla^2 v &= 2k\,\frac{\partial T}{\partial y} \\
e &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\end{aligned}
\tag{E5-4.3}
$$

Let

$$
u = \frac{\partial \psi}{\partial x}, \qquad v = \frac{\partial \psi}{\partial y} \tag{E5-4.4}
$$

where $\psi = \psi(x, y)$ is called the displacement potential function. Substitution of Eqs. (E5-4.4) into Eqs. (E5-4.3) yields

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1}{1+\nu}\nabla^2\psi - kT\right) &= 0 \\
\frac{\partial}{\partial y}\left(\frac{1}{1+\nu}\nabla^2\psi - kT\right) &= 0
\end{aligned}
$$

These equations are satisfied identically if

$$\nabla^2\psi = (1+\nu)kT \tag{E5-4.5}$$

Accordingly, the solution of Eq. (E5-4.5) represents a particular solution of Eqs. (E5-4.3). To obtain a general solution of Eqs. (E5-4.3), we must add to the solution of Eq. (E5-4.5) the complementary solution of Eqs. (E5-4.3); that is, we must add the solution of Eqs. (E5-4.3) for the case $T = 0$. This general solution must then be made to satisfy the boundary conditions of the problem.

By Eqs. (E5-4.2), (E5-4.4), and (E5-4.5), the stress components corresponding to the particular solution ψ are

$$\begin{aligned} \sigma'_x &= -2G\,\frac{\partial^2\psi}{\partial y^2} \\ \sigma'_y &= -2G\,\frac{\partial^2\psi}{\partial x^2} \\ \tau'_{xy} &= 2G\,\frac{\partial^2\psi}{\partial x\,\partial y} \end{aligned} \tag{E5-4.6}$$

In the absence of temperature T, the complementary solution of the plane problem is expressed in terms of the Airy stress function F [Eqs. (5-4.9)]. Accordingly, the stress components for a general solution of the plane stress thermoelastic problem are, by Eqs. (5-4.9) and (E5-4.6),

$$\begin{aligned} \sigma_x &= \frac{\partial^2}{\partial y^2}(F - 2G\psi) \\ \sigma_y &= \frac{\partial^2}{\partial x^2}(F - 2G\psi) \\ \tau_{xy} &= -\frac{\partial^2}{\partial x\,\partial y}(F - 2G\psi) \end{aligned} \tag{E5-4.7}$$

Similarly, for the case of plane strain, we have

Stress–Displacement Relations:

$$\begin{aligned} \sigma_x &= \lambda e + 2G\,\frac{\partial u}{\partial x} - \frac{EkT}{1-2\nu} \\ \sigma_y &= \lambda e + 2G\,\frac{\partial v}{\partial y} - \frac{EkT}{1-2\nu} \\ \sigma_z &= \nu(\sigma_x + \sigma_y) - EkT = \lambda e - \frac{EkT}{1-2\nu} \\ \tau_{xy} &= G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad G = \frac{E}{2(1+\nu)}, \qquad e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{aligned} \tag{E5-4.8}$$

Equilibrium Equations in Terms of Displacement:

$$\begin{aligned}\frac{\partial e}{\partial x} + (1-2\nu)\nabla^2 u &= 2(1+\nu)\,\frac{\partial(kT)}{\partial x}\\ \frac{\partial e}{\partial y} + (1-2\nu)\nabla^2 v &= 2(1+\nu)\,\frac{\partial(kT)}{\partial y}\end{aligned} \qquad \text{(E5-4.9)}$$

Displacement Potential–Temperature Relation:

$$\nabla^2\psi = \frac{1+\nu}{1-\nu}\,kT \qquad \text{(E5-4.10)}$$

With the displacement potential function ψ defined by Eq. (E5-4.10), the stress components (σ_x, σ_y, τ_{xy}) are again given by Eqs. (E5-4.7). Then σ_z is determined by Eq. (E5-4.8).

In the preceding method of integration of the stress equations we have used a stress function or a displacement potential. In a certain class of problems the thermal-stress equations may be integrated more directly by other methods (Sen, 1939; Sharma, 1956; McDowell and Sternberg, 1957).

Problem Set 5-4

1. In a state of plane strain relative to the (x, y) plane, the displacement component $w = 0$ and the displacement components (u, v) are functions of (x, y) only. Hence, the components of rotation $\omega_x = \omega_y = 0$ and $\omega = \omega_z$. For zero body forces (set $V = 0$), we note that the equations of equilibrium are satisfied by Eqs. (5-4.9). Show that

$$\sigma_x + \sigma_y = 2(\lambda + G)e$$

where e is the volumetric strain or dilatation and where λ, G are the Lamé constants. Hence, show that in terms of dilatation and rotation the equations of equilibrium are

$$(\lambda + 2G)\,\frac{\partial e}{\partial x} - 2G\,\frac{\partial \omega}{\partial y} = 0, \qquad (\lambda + 2G)\,\frac{\partial e}{\partial y} + 2G\,\frac{\partial \omega}{\partial x} = 0$$

Thus, show that e and ω are plane harmonic functions.

2. Because the dilatation e and rotation ω are plane harmonic functions (see Problem 1), $(\lambda + 2G)e + i2G\omega$ is a function of the complex variable $x + iy$, where i is $\sqrt{-1}$. Also, the Airy stress function F is related to e by $\nabla^2 F = 2(\lambda + G)e$, where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Introduce the new function $\xi + i\eta$ of $x + iy$ as follows:

$$\xi + i\eta = \int [(\lambda + 2G)e + i2G\omega]\, d(x + iy)$$

so that

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = (\lambda + 2G)e = \frac{\lambda + 2G}{2(\lambda + G)} \nabla^2 F, \qquad -\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = 2G\omega$$

where F is Airy's stress function. Hence, show that

$$2G\,\frac{\partial u}{\partial x} = \frac{\partial^2 F}{\partial y^2} - \frac{\lambda}{2(\lambda + G)} \nabla^2 F = -\frac{\partial^2 F}{\partial x^2} + \frac{\partial \xi}{\partial x}$$

$$2G\,\frac{\partial v}{\partial y} = \frac{\partial^2 F}{\partial x^2} - \frac{\lambda}{2(\lambda + G)} \nabla^2 F = -\frac{\partial^2 F}{\partial y^2} + \frac{\partial \eta}{\partial y}$$

and that

$$2G\,\frac{\partial u}{\partial y} = -\frac{\partial^2 F}{\partial x\,\partial y} - 2G\omega = -\frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial \xi}{\partial y}$$

$$2G\,\frac{\partial v}{\partial x} = -\frac{\partial^2 F}{\partial x\,\partial y} + 2G\omega = -\frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial \eta}{\partial x}$$

and that there follows

$$2Gu = -\frac{\partial F}{\partial x} + \xi, \qquad 2Gv = -\frac{\partial F}{\partial y} + \eta$$

These equations define the displacement components (u, v) when F is known.

3. We recall that

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \qquad 2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

These equations with the definitions of ξ, η given in Problem 2 yield, after integration,

$$u = \frac{\partial}{\partial x}\left[\frac{y\eta}{2(\lambda + 2G)}\right] + \frac{\partial}{\partial y}\left[\frac{y\xi}{2G}\right] + u'$$

$$v = \frac{\partial}{\partial y}\left[\frac{y\eta}{2(\lambda + 2G)}\right] - \frac{\partial}{\partial x}\left[\frac{y\xi}{2G}\right] + v'$$

where $v' + iu'$ is a function of $x + iy$. Let $u' = \partial f/\partial x$, $v' = \partial f/\partial y$, $\nabla^2 f = 0$. Show that

$$u = \frac{\xi}{2G} + \frac{\lambda + G}{2G(\lambda + 2G)} y\,\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial x}$$

$$v = \frac{\eta}{2(\lambda + 2G)} - \frac{\lambda + G}{2G(\lambda + 2G)} y\,\frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial y}$$

These equations define (u, v) when e and ω are known.

4. With the information given in Problems 2 and 3, show that

$$F = -2Gf + \frac{\lambda + G}{\lambda + 2G} y\eta$$

and that the formulas for (u, v) given in Problems 2 and 3 are thus equivalent.

5. Let a thin plate with constant thickness and with mass density ρ rotate with constant angular velocity ω above the y axis (Fig. P5-4.5). Neglecting gravity, write an expression for the inertia force X per unit volume (body force per unit volume) that acts on an arbitrary mass element of the plate. Write the differential equations of equilibrium for the plate. Write the general solution of these equations in terms of Airy's stress function F. Show that the equation of compatibility is $\nabla^4 F = (1 - \nu)\rho\omega^2$, where ν is Poisson's ratio.

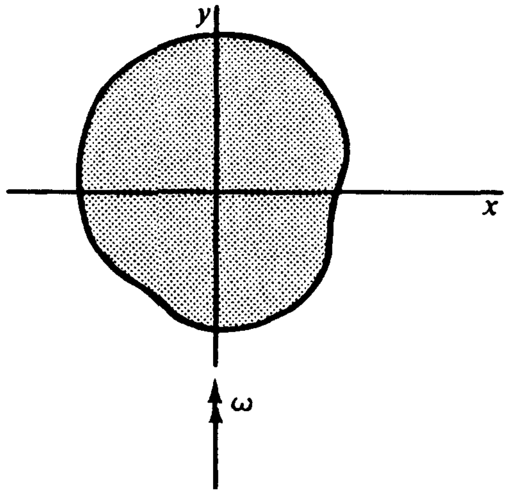


Figure P5-4.5

6. An infinite plane strip is bounded by the lines $y = \pm 1$. The stresses on the lines $y = \pm 1$ are $\sigma_y = \cos x$, $\tau_{xy} = 0$. There is no body force. By assuming an Airy stress function of the form $f(y)\cos x$, determine σ_x, σ_y, τ_{xy} as functions of (x, y).

7. The following stress–strain relations pertain to the anisotropic flat thin plate subjected to a state of generalized plane stress:

$$\epsilon_x = S_{11}\sigma_x + S_{12}\sigma_y$$
$$\epsilon_y = S_{12}\sigma_x + S_{22}\sigma_y$$
$$\gamma_{xy} = S_{33}\tau_{xy} \quad (x, y) = \text{rectangular Cartesian coordinates}$$

where S_{11}, S_{22}, S_{33}, S_{12} are elastic constants and where $(\sigma_x, \sigma_y, \tau_{xy})$ and $(\epsilon_x, \epsilon_y, \gamma_{xy})$ are average values of stress and strain through the thickness. Let $(\sigma_x, \sigma_y, \tau_{xy})$ be defined in terms of an Airy stress function F. Show that the defining equation for the Airy stress function F is of the form

$$\left(\frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 F}{\partial x^2} + \alpha_2 \frac{\partial^2 F}{\partial y^2}\right) = 0 \tag{a}$$

where α_1, α_2 are constants. For the case $S_{11} = S_{22} = 1/E$, $S_{12} = -\nu/E$, $S_{33} = 1/G$, show that Eq. (a) reduces to the biharmonic equation.

8. Let

$$F = ax^2 + by^3 + \sum_{n=1}^{\infty} A_n(y)\cos\left(\frac{n\pi x}{L}\right)$$

be an Airy stress function for a plane, isotropic problem, where a, b, L are constants, and $A_n(y)$ are functions of y, Derive the defining differential equation for the coefficients A_n.

Consider a plane rectangular region $-L \le x \le L$, $-C \le y \le C$. Assume that no net force or no net couple acts on the sections $x = \pm L$. Discuss how the arbitrary constants in the solution of the differential equation for $A_n(y)$ may be evaluated.

9. Consider a case of plane stress without body forces in the region $-c \le y \le c$, $0 \le x \le \ell$ (see Fig. P5-4.9). If the resultant of the stresses in the x direction is zero, the elementary beam formula yields $\sigma_x = My/I$; that is, σ_x is a linear function of y.

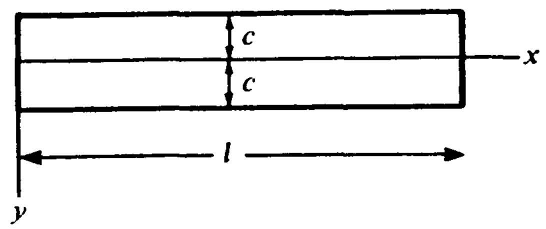


Figure P5-4.9

(a) Let $\sigma_x = F_{yy}$, $\sigma_y = F_{xx}$, $\tau_{xy} = -F_{xy}$. Write the most general expression for $F(x, y)$ that satisfies the equations of equilibrium and yields σ_x as linear function of y in the form $\sigma_x = yf(x)$.

(b) Assuming that the material is isotropic and linearly elastic, write the equation of compatibility for $F(x, y)$ as determined in part (a).

(c) Determine the most general form of $F(x, y)$ that satisfies the equations of equilibrium and compatibility, and yields σ_x linear in y.

(d) Derive expressions for the stress components using the stress function derived in part (c).

(e) Assume that no load is applied along the line $y = c$. Show that the elementary formula can be correct, strictly speaking, only if the stresses are those produced in a cantilever with a concentrated vertical load at the end and/or a moment applied at the end.

10. The general stress–strain–temperature relationship for an isotropic material is

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{v}{E}\sigma_y - \frac{v}{E}\sigma_z + kT$$

$$\epsilon_y = -\frac{v}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{v}{E}\sigma_z + kT$$

$$\epsilon_z = -\frac{v}{E}\sigma_x - \frac{v}{E}\sigma_y + \frac{1}{E}\sigma_z + kT$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}, \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}, \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Consider a body that is in a state of plane strain.

(a) Derive the "two-dimensional" Hooke's law expressing the strains ϵ_x, $\epsilon_y \ldots$ as functions of σ_x, σ_y, τ_{xy}, and $T = T(x, y)$.

(b) Assuming that body forces are negligible, let $\sigma_x = F_{yy}$, $\sigma_y = F_{xx}$, $\tau_{xy} = -F_{xy}$, where F is a stress function. Derive the compatibility conditions in terms of T and F. Thus, show that $F(x, y)$ must be biharmonic if $T(x, y)$ is harmonic.

11. For a plane problem, the stress components in the (x, y) rectangular region $0 \leq x \leq L$, $-c \leq y \leq c$, where L and c are constants, are given by the relations (q = constant)

$$\sigma_x = \frac{qx^3y}{4c^3} + \frac{q}{4c^3}\left(-2xy^3 + \frac{6}{5}c^2xy\right)$$

$$\sigma_y = -\frac{qx}{2} + qx\left(\frac{y^3}{4c^3} - \frac{3y}{4c}\right)$$

$$\tau_{xy} = \frac{3qx^2}{8c^3}(c^2 - y^2) - \frac{q}{8c^3}(c^4 - y^4) + \frac{q}{4c^3} \cdot \frac{3c^2}{5}(c^2 - y^2)$$

(a) Show that these stress components satisfy the equations of equilibrium in the absence of body forces.

(b) Derive the Airy stress function from which these stress components are derivable.

(c) Show that the stress state is compatible.

(d) Determine the problem that the stress components represent.

12. The stress function for a cantilever beam loaded by a shear force P at the free end is

$$F = C_1xy^3 + C_2xy$$

(a) Evaluate the constants C_1 and C_2.

(b) Derive the expressions for the displacements u and v.

(c) Compare v with the expression derived for displacement y from elementary beam theory, $EI(d^2y/dx^2) = M$.

13. Apply the stress function $F = -(P/d^3)xy^2(3d - 2y)$ to the region $0 \leq y \leq d$, $0 \leq x$. Determine what kind of problem is solved by this stress function.

14. The stress–strain relationship for a certain orthotropic material may be written as

$$\epsilon_\alpha = C_{\alpha\beta}\sigma_\beta, \qquad \alpha, \beta = 1, 2, \ldots, 6$$

where

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

and

$$\begin{array}{llllll} \sigma_1 = \sigma_x, & \sigma_z = \sigma_y, & \sigma_3 = \sigma_z, & \sigma_4 = \tau_{xy}, & \sigma_5 = \tau_{xz}, & \sigma_6 = \tau_{yz} \\ \epsilon_1 = \epsilon_x, & \epsilon_2 = \epsilon_y, & \epsilon_3 = \epsilon_z, & \epsilon_4 = \gamma_{xy}, & \epsilon_5 = \gamma_{xz}, & \epsilon_6 = \gamma_{yz} \end{array}$$

(a) For this material derive the fourth-order partial differential equation that a stress function must satisfy in order to meet equilibrium and compatibility requirements for plane stress in the xy plane. Neglect body forces.

(b) Show that the equation derived in part (a) reduces to $\nabla^4 F = 0$ for an isotropic material.

15. Consider the Airy stress function $F = Ax^3y$, where A is a constant and (x, y) are rectangular Cartesian coordinates. Determine the plane elasticity problem that is solved by this function for the region $-a \leq x \leq a$, $-b \leq y \leq b$.

16. Show that the function

$$F = \frac{q}{20c^3}[10x^2(2y^3 - 3cy^2) - 2y^2(2y^3 - 5cy^2 + 4c^2y - c^3)]$$

may be employed as a stress function. For the plane region $0 \leq x \leq L$, $0 \leq y \leq c$, determine the stress boundary conditions, and describe fully the plane problem for which the stress function serves as the solution for equilibrium.

17. Show that the three-dimensional equilibrium equations without body force are satisfied, if the stresses are derived from any six functions A, B, C, L, M, N as follows

$$\begin{array}{ll} \sigma_x = B_{zz} + C_{yy} - 2L_{yz}, & \tau_{yz} = -A_{yz} + (M_y + N_z - L_x)_x \\ \sigma_y = C_{xx} + A_{zz} - 2M_{zx}, & \tau_{zx} = -B_{zx} + (N_z + L_x - M_y)_y \\ \sigma_z = A_{yy} + B_{xx} - 2N_{xy}, & \tau_{xy} = -C_{xy} + (L_x + M_y - N_z)_z \end{array}$$

Subscripts on A, B, C, L, M, N denote partial derivatives.

By discarding some of the above functions, obtain Airy's solution to the equilibrium equations of plane stress theory relative to the yz plane.

18. A dam or retaining wall is subjected to a linearly varying pressure $p = p_0y$. The slice shown in Fig. P5-4.18 is assumed to be in a plane state, with all quantities functions of (x, y) only.

(a) Write down the stress boundary conditions for the faces of AO, BO.

(b) On the basis of part (a), write the simplest Airy stress function that will ensure satisfaction of the boundary conditions on AO, BO. Explain your choice.

(c) Let the body force of the dam be ρg in the y direction, where ρ is the mass density and g is the gravity acceleration. Including the effect of body forces, determine explicitly in terms of known quantities the complete expressions for σ_x, σ_y, τ_{xy}. (*Hint*: Note that the body force ρg is derivable from the potential function $V = -\rho gy$.)

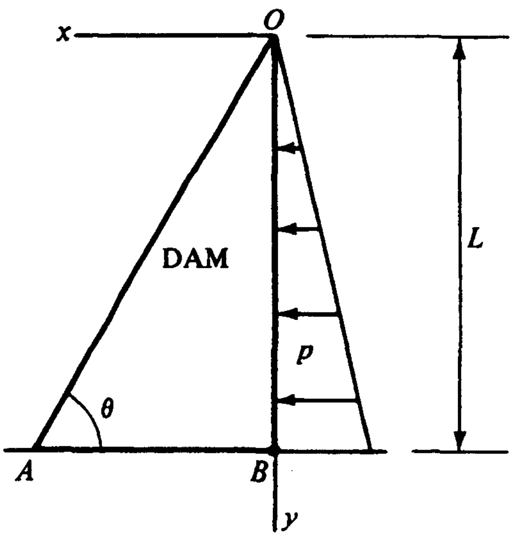


Figure P5-4.18

19. A solution to a plane strain equilibrium problem in the absence of body forces is generated by the Airy stress function Ayx^3.

(a) Determine whether this solution is compatible for a three-dimensional problem.

(b) With this Airy stress function, derive expressions for the stress components; hence, for a linearly elastic isotropic material, derive the corresponding strain components.

20. For homogeneous orthotropic plane stress problems, the stress–strain relations relative to (x, y) axes are:

$$\begin{aligned} \epsilon_x &= (\sigma_x/E_x) - \nu_{xy}(\sigma_y/E_y); \qquad \epsilon_y = (\sigma_y/E_y) - \nu_{yx}(\sigma_x/E_x) \\ \gamma_{xy} &= \tau_{xy}/G = \{[E_x + (1 + 2\nu_{yx})E_y]/E_xE_y\}\tau_{xy} \end{aligned} \tag{a}$$

where the symbols are self-explanatory. The strain energy density U is given by the formula

$$U + A\epsilon_x^2 + B\epsilon_y^2 + 2C\epsilon_x\epsilon_y + D\gamma_{xy}^2 \tag{b}$$

(a) By the relations $\sigma_\alpha = \partial U/\partial \epsilon_\alpha$, derive the stress–strain relations.

(b) With the result of part (a) and Eq. (a), derive a relationship among E_x, ν_{xy}, E_y, and ν_{yx}.

(c) As in the isotropic case, assume that a stress function $F(x, y)$ exists such that

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y} \tag{c}$$

Derive the defining equation for the stress function $F(x, y)$ in the form

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) = 0 \tag{d}$$

where k^2 is expressed in terms of E_x, E_y.

(d) Let $F = (P/6I)(3b^2xy - xy^3)$, where P, B, and I are constants. Show that F satisfies the equation $\nabla^2\nabla^2 F = 0$ for isotropic materials and also satisfies Eq. (d). Hence, F is an appropriate stress function for both isotropic and orthotropic materials.

5-5 Airy Stress Function in Terms of Harmonic Functions

In this section we consider the problem of representation of the Airy stress function in terms of a pair of suitably chosen conjugate harmonic functions and a third harmonic function. Such a representation allows us to express the general solution of the biharmonic equation in terms of harmonic functions.

Let ϕ be a harmonic function in (x, y); that is, $\nabla^2\phi = 0$, where ∇^2 is the two-dimensional Laplacian. It may be shown that a solution of the biharmonic equation $\nabla^2\nabla^2 F = 0$ may be expressed in terms of ϕ by any one of the following forms:

$$x\phi, \qquad y\phi, \qquad (x^2 + y^2)\phi \tag{5-5.1}$$

We note that a function Q_1 defined by

$$Q_1 = \nabla^2 F = \sigma_x + \sigma_y \tag{5-5.2}$$

where F is the Airy stress function, is harmonic in the absence of body forces and temperature, as $\nabla^2 Q_1 = \nabla^2\nabla^2 F = 0$. The function Q_2 related to Q_1 by the Cauchy–Riemann equations (Churchill et al., 1989)

$$\frac{\partial Q_1}{\partial x} = \frac{\partial Q_2}{\partial y}, \qquad \frac{\partial Q_1}{\partial y} = -\frac{\partial Q_2}{\partial x} \tag{5-5.3}$$

is the conjugate harmonic of Q_1. By Eqs. (5-5.2) and (5-5.3), we note that

$$\nabla^2 Q_1 = \nabla^2 Q_2 = 0 \tag{5-5.4}$$

That is, Q_2 is harmonic.

By the Cauchy integral theorem (Churchill et al., 1989) of complex variables, the integral of the analytic function

$$f(z) = Q_1 + iQ_2 \tag{5-5.5}$$

where $z = x + iy$, $i = \sqrt{-1}$, is another analytic function, say $\psi(z)$. Thus,

$$\psi(z) = q_1 + iq_2 = \frac{1}{c}\int f(z)\, dz \tag{5-5.6}$$

is analytic, where c is as yet an arbitrary constant. The functions (q_1, q_2) are conjugate harmonic functions; that is, they satisfy Eqs. (5-5.3). We note by Eq.

(5-5.6) that $\psi'(z) = (1/c)f(z)$, where the prime denotes differentiation with respect to z. Hence,

$$\frac{\partial q_1}{\partial x} + i\,\frac{\partial q_2}{\partial x} = \frac{\partial}{\partial x}\,\psi(z) = \frac{\partial \psi}{\partial z}\,\frac{\partial z}{\partial x}$$

Because $\partial z/\partial x = 1$, we obtain from the above results and Eq. (5-5.5)

$$\frac{\partial q_1}{\partial x} + i\,\frac{\partial q_2}{\partial x} = \frac{1}{c}(Q_1 + iQ_2) \tag{5-5.7}$$

Equating real parts of Eq. (5-5.7), we obtain

$$\frac{\partial q_1}{\partial x} = \frac{1}{c}Q_1 \tag{5-5.8}$$

Because (q_1, q_2) satisfy Eqs. (5-5.3), we obtain from Eqs. (5-5.3) and (5-5.8)

$$\frac{\partial q_2}{\partial y} = \frac{1}{c}Q_1 \tag{5-5.9}$$

Accordingly, by Eqs. (5-5.8), (5-5.9), and (5-5.2), we find that p_0 defined by

$$p_0 = F - xq_1 - yq_2 \tag{5-5.10}$$

is harmonic, provided $c = 4$. Accordingly, the Airy stress function F may be written in the form

$$F = xq_1 + yq_2 + p_0 \tag{5-5.11}$$

where (q_1, q_2) are suitably chosen conjugate harmonic functions and p_0 is an arbitrary harmonic function. Alternatively, we may take F in the forms (provided $c = 4$)

$$F = 2xq_1 + p_1 \tag{5-5.12}$$

or

$$F = 2yq_2 + p_2 \tag{5-5.13}$$

where (p_1, p_2) are arbitrary harmonic functions.

5-6 Displacement Components for Plane Elasticity

Direct Integration Method. When the plane elasticity stress components σ_x, σ_y, τ_{xy} are known, the strain components ϵ_x, ϵ_y, γ_{xy} may be determined by Eqs. (5-3.6) for generalized plane stress or by Eqs. (5-3.2) for plane strain. Then integration of the strain–displacement relations [Eqs. (5-1.4) for plane strain or Eqs. (5-1.4) with ϵ_z given by Eqs. (5-3.4) for generalized plane stress] yields the (x, y) displacement components (u, v). The integration of the strain–displacement relations yields an arbitrary rigid-body displacement (see Section 2-15 in Chapter 2 and Examples 4-18.1 and 4-18.2 in Chapter 4). Accordingly, complete specification of the displacement (u, v) requires that the rigid-body displacement of the body be known. For example, in Example 4-18.1, it was specified that the point $x = y = z = 0$ be fixed and that the volumetric rotation for this point vanish. Consequently, the displacements and rotations of all other points and volume elements in the body were determined relative to the point and volume element at $x = y = z = 0$. Similarly, to fix the rigid-body displacement in the solution of the plane problem, we may specify the displacement of some point (say, x_0, y_0) and the rotation of a line element (say, a line element through point x_0, y_0).

Representation in Terms of Airy Stress Function. Alternatively, we may derive formulas for the plane displacement components (u, v) in terms of the Airy stress function. We carry out the calculation for the case of the plane stress. The results for plane strain may be obtained in a similar manner.

For plane stress relative to the (x, y) plane, the stress–strain relations are

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \epsilon_y &= \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{G}\tau_{xy} \end{aligned} \tag{5-6.1}$$

where $(\epsilon_x, \epsilon_y, \gamma_{xy})$ are the strain components, $(\sigma_x, \sigma_y, \tau_{xy})$ are stress components, (u, v) denote the (x, y) displacement components, E denotes the modulus of elasticity, ν is Poisson's ratio, and $G = E/[2(1 + \nu)]$.

In terms of the Airy stress function F, the stress components are [Eqs. (5-4.9) with $V = 0$]

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y} \tag{5-6.2}$$

Equations (5-6.1) and (5-6.2) yield [with Eq. (5-5.2)]

$$\begin{aligned} E\,\frac{\partial u}{\partial x} &= -(1 + \nu)\,\frac{\partial^2 F}{\partial x^2} + Q_1 \\ E\,\frac{\partial v}{\partial y} &= -(1 + \nu)\,\frac{\partial^2 F}{\partial y^2} + Q_1 \end{aligned} \tag{5-6.3}$$

We replace Q_1 by $4(\partial q_1/\partial x)$ in the first of Eqs. (5-6.3) and by $4(\partial q_2/\partial y)$ in the second [see Eqs. (5-5.8) and (5-5.9)]. Thus, after dividing by $1+\nu$, we find

$$\begin{aligned} 2G\frac{\partial u}{\partial x} &= -\frac{\partial^2 F}{\partial x^2} + \frac{4}{1+\nu}\frac{\partial q_1}{\partial x} \\ 2G\frac{\partial v}{\partial y} &= -\frac{\partial^2 F}{\partial y^2} + \frac{4}{1+\nu}\frac{\partial q_2}{\partial y} \end{aligned} \tag{5-6.4}$$

Integration of Eqs. (5-6.4) yields

$$\begin{aligned} 2Gu &= -\frac{\partial F}{\partial x} + \frac{4}{1+\nu}q_1 + f_1(y) \\ 2Gv &= -\frac{\partial F}{\partial y} + \frac{4}{1+\nu}q_2 + f_2(x) \end{aligned} \tag{5-6.5}$$

where $f_1(y), f_2(x)$ are arbitrary functions of integration.

To interpret (f_1, f_2) of Eqs. (5-6.5), we note that by the last of Eqs. (5-6.2) and Eqs. (5-6.5), with Eqs. (5-5.3)

$$\tau_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{\partial^2 F}{\partial x\,\partial y} + \frac{1}{2}\frac{df_1}{dy} + \frac{1}{2}\frac{df_2}{dx}$$

Hence,

$$\frac{df_1}{dy} + \frac{df_2}{dx} = 0 \tag{5-6.6}$$

Integration of Eq. (5-6.6) yields

$$f_1 = Ay + B, \qquad f_2 = -Ax + C \tag{5-6.7}$$

Hence, the functions (f_1, f_2) represent a rigid-body displacement (Section 2-15 in Chapter 2). Discarding them, we get

$$\begin{aligned} 2Gu &= -\frac{\partial F}{\partial x} + \frac{4}{1+\nu}q_1 \\ 2Gv &= -\frac{\partial F}{\partial y} + \frac{4}{1+\nu}q_2 \end{aligned} \tag{5-6.8}$$

Equations (5-6.8) determine displacement components (u, v) when the stress function F is known. The function Q_1 is determined by computing $\nabla^2 F$ [Eq. (5-5.2)]. Then the function Q_2 is determined by means of the Cauchy–Riemann equations [Eqs. (5-5.3)]. The functions (q_1, q_2) are then determined by integration of the function $f(z) = Q_1 + iQ_2$ [Eqs. (5-5.5) and (5-5.6)].

The method outlined above is useful for the determination of displacement components (u, v) for those cases in which direct integration of the strain–displacement relations fails (see Examples 4-18.1 and 4-18.2).

Example 5-6.1. Stress Function for the Flexural Wrinkling of a Sandwich Panel. Because of in-plane compressive forces $(\bar{F})$ in the compression facing of a sandwich panel (Fig. E5-6.1), flexural wrinkling (Chong and Hartsock, 1974), which is a localized instability, may occur prior to overall buckling. The compression facing can be treated approximately as a plate supported by the elastic core bounded by the tension facing. The core under plane strain conditions is governed by Eq. (5-4.12) in terms of the Airy stress function F:

$$\frac{\partial^4 F}{\partial x^4} + 2\,\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{a}$$

Equation (a) may be satisfied by taking F in the form (Timoshenko and Goodier, 1970)

$$F = F(x, y) = f(y)\sin \alpha x \tag{b}$$

provided $f(y)$ satisfies the equation

$$\frac{\partial^4 f}{\partial y^4} - 2\alpha^2\,\frac{\partial^2 f}{\partial y^2} + \alpha^4 f = 0 \tag{c}$$

The solution to Eq. (c) is

$$f(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y \tag{d}$$

To determine C_1, C_2, C_3, and C_4, the following four boundary conditions are used:

$$\text{At}\quad y = 0:\qquad \sigma_y = -q_m \sin \alpha x; \qquad \epsilon_x = 0 \tag{e}$$

$$\text{At}\quad y = D:\qquad \epsilon_x = 0; \qquad \frac{\partial v}{\partial x} = 0 \tag{f}$$

where q_m is the amplitude of the stress at the interface resulting from deformation of the compression facing. Expressing the stresses, hence the strain ϵ_x, in terms of the

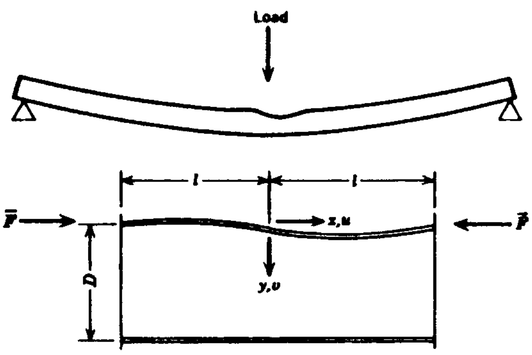


Figure E5-6.1

Airy stress function, we may employ Eqs. (e) and (f) to determine the constants C_1, C_2, C_3, and C_4. The resulting stress function F is

$$F(x, y) = \frac{q_m}{\alpha^2} \sin \alpha x \left\{ \cosh \alpha y - \frac{\alpha y}{2(1-v)} \sinh \alpha y \right.$$
$$- [(1+v)\beta^2 + \sinh^2 \beta(-6 + 8v + 6v^2 - 8v^3)]$$
$$\left. \times \frac{\alpha}{2(1-v)\Delta} \sinh \alpha y - [\alpha^2 \sinh^2 \ \beta(3 - v - 4v^2)] \frac{y \cosh \alpha y}{2(1-v)\Delta} \right\} \qquad \text{(g)}$$

in which $\Delta = -\alpha[(1+v)\beta + (3 - v - 4v^2) \sinh \beta \cosh \beta]$, and $\beta = \alpha D$.

Problem Set 5-6

1. The skewed plate of unit thickness is loaded by uniformly distributed stresses S_1 and S_2 applied perpendicularly to the sides of the plate (see Fig. P5-6.1).
   (a) Determine all conditions of equilibrium for the plate in terms of S_1, S_2, a, b, and θ.
   (b) For $\theta = 90°$, derive an expression for the elongation of the diagonal AC under the action of S_1 and S_2. Assume that the material is homogeneous, isotropic, and linearly elastic, and that the displacements are small.

2. In Fig. P5-6.1, let S_1 and S_2 be applied so that they are directed parallel to the edges $AB(DC)$ and $AD(BC)$ of the skewed plate. Assuming that the plate is elastic, derive expressions for the principal stresses and the principal strains in terms of S_1, S_2, a, b, θ, E, and v, where E and v denote Young's modulus and Poisson's ratio, respectively.

3. Let isotropic elastic material in the (x, y) plane be subjected to the stress components $\sigma_x = 0$, $\sigma_y = \sigma$, $\tau_{xy} = \tau$. Let $u = v = \omega = 0$ for $x = y = 0$, where (u, v) denote (x, y) displacement components and ω denotes volumetric rotation.
   (a) Show that the circle $x^2 + y^2 = a^2$ is deformed into an ellipse.
   (b) For the case $\tau = 0$, show that the major and minor axes of the ellipse coincide with the (x, y) axes, and express their lengths in terms of a and the elastic properties of the material.

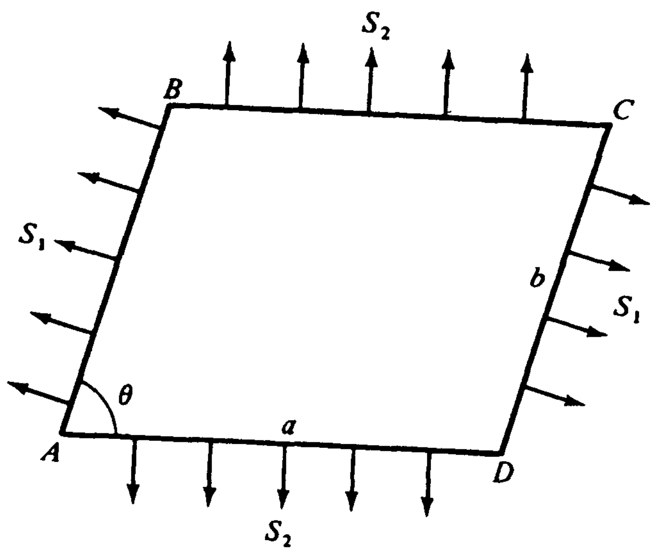


Figure P5-6.1

4. For the isotropic, homogeneous, and elastic cantilever beam shown in Fig. P5-6.4, the stresses are given by

$$\sigma_x = \frac{P}{I}(L - x)y, \qquad \tau_{xy} = \frac{P}{2I}(y^2 - c^2), \qquad \sigma_y = 0$$

where P, I, L, and c are constants.

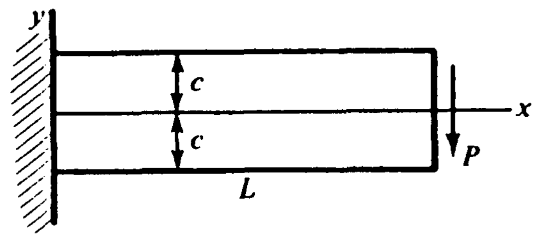


Figure P5-6.4

(a) Verify that these stresses satisfy equilibrium and compatibility conditions for plane stress.

(b) Determine the strains, hence the displacements u and v, as functions of x and y. The boundary conditions are for $x = y = 0$, $u = v = 0$, and an infinitesimal line segment originally in the y direction does not rotate.

5. The rectangular plate shown in Fig. P5-6.5 is very thin in the z direction and has a length in the $\pm x$ directions that is very large compared to $2a$. The plate is made of a nonlinear

elastic isotropic homogeneous material whose stress–strain relations are

$$\epsilon_x = A\sigma_x^3 - B\sigma_y^3, \qquad \epsilon_y = A\sigma_y^3 - B\sigma_x^3$$

where A and B are known constants. The plate is subjected to angular velocity ω about the x axis. The mass density of the plate is ρ. Assume that $\tau_{xy} = u = X = \partial/\partial x = 0$. Determine the stresses σ_x and σ_y and the displacement v as functions of y.

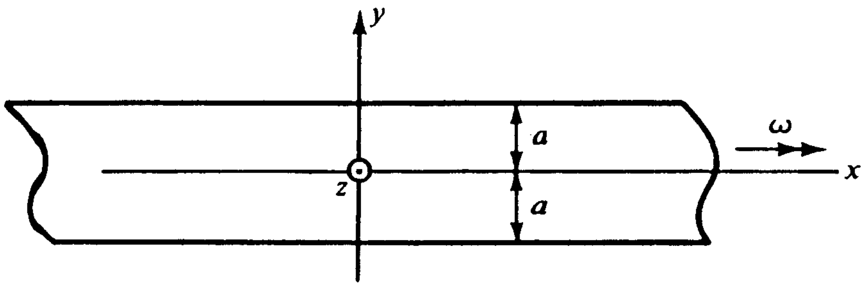


Figure P5-6.5

6. A narrow uniform bar of density ρ and length $2b$ is rotating with an angular velocity ω about an axis perpendicular to the bar through its center. Neglecting gravity effects and assuming linear elastic behavior, determine the increase in length of the bar resulting from the rotation.

7. Assume the plate of Prob. 5 is linearly elastic. Determine the stress components σ_x, σ_y and the displacement component v as functions of y.

8. Consider the equations of linear elasticity of a homogeneous isotropic body. For example, the equations of motion are

$$(\lambda + G)\,\frac{\partial^2 u_\alpha}{\partial x_\alpha\,\partial x_\beta} + G\,\frac{\partial^2 u_\beta}{\partial x_\alpha\,\partial x_\alpha} = \rho\,\frac{\partial^2 u_\beta}{\partial t^2}$$

For the case of static equilibrium, assume that u_α is representable in the form

$$2Gu_\alpha = \frac{\partial \phi}{\partial x_\alpha}$$

where ϕ is a scalar function of rectangular Cartesian coordinates (x_1, x_2, x_3).

(a) Derive the defining equation for ϕ.

(b) In terms of ϕ, derive expressions for the volumetric strain (dilatation) e, the strain tensor (small displacement) $\epsilon_{\alpha\beta}$, and the stress tensor $\sigma_{\alpha\beta}$.

(c) Let $F = A(x^2 - y^2) + 2Bxy$ be an Airy stress function, where (A, B) are constants and (x, y) are plane rectangular Cartesian coordinates. Determine the problem solved by this function F for the plane rectangular region $-a \le x \le a$, $-b \le y \le b$.

9. The thin homogeneous plane strip of width $2h$ extends a great distance in the $\pm x$ direction Fig. P5-6.9. The plate is rigidly restrained by the fixed walls at $y = \pm h$. The plate is loaded by gravity in the $-y$ direction. The density of the plate is ρ. Assume $\partial/\partial x = u = \tau_{xy} = X = 0$. The plate is made of a material whose stress–strain relations are

$$\epsilon_x = A\sigma_x^3 - B\sigma_y^3, \qquad \epsilon_y = A\sigma_y^3 - B\sigma_x^3, \qquad \gamma_{xy} = C\tau_{xy}$$

where A, B, and C are known constants. Determine formulas for σ_x, σ_y, and v as functions of y and the known constants A, B, C, ρ, g, and h.

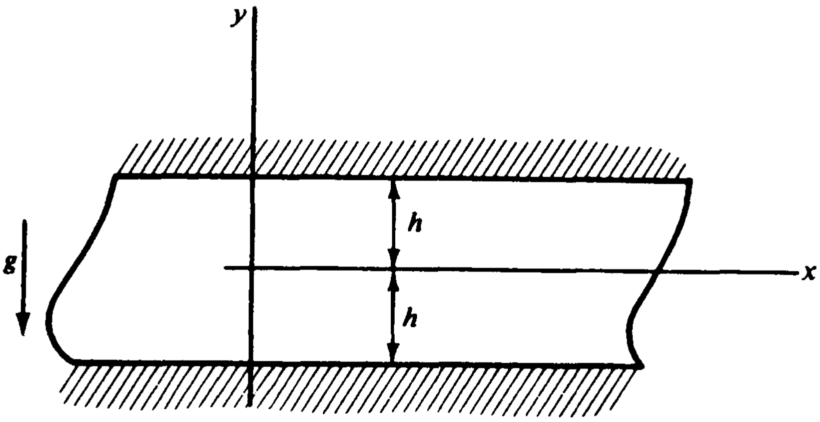


Figure P5-6.9

10. A flat strip is supported at one end ($x = 0$, Fig. P5-6.10) and hangs in a gravity field of acceleration g. The mass density of the strip is ρ. Let the thickness of the strip be 1 unit.

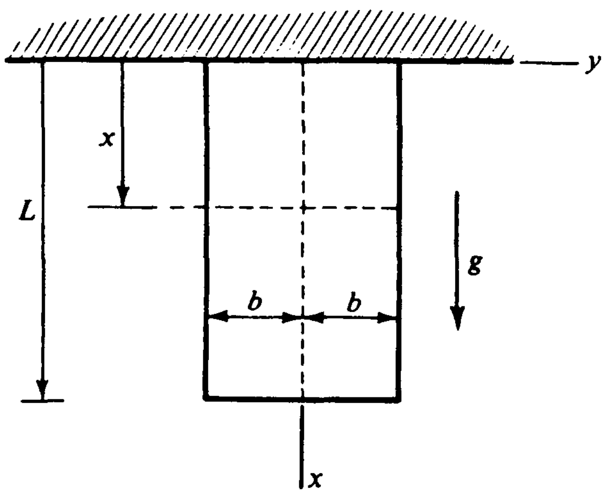


Figure P5-6.10

Assume a state of plane stress relative to the (x, y) plane.

(a) Consider the equilibrium of the part of the bar from $x = x$ to $x = L$. Write expressions for the body forces X, Y and for the net force acting at section x.

(b) Assume the simplest possible stress distribution in the bar and derive an expression for the normal stress σ_x.

(c) By the semi-inverse method, determine whether equations of elasticity are satisfied by the results of parts (b) and (a).

(d) Derive explicit expressions for the (x, y) displacement components (u, v) in terms of properties of the bar and (x, y). Let $u = v = \partial u/\partial y = 0$ at $x = y = 0$.

5-7 Polynomial Solutions of Two-Dimensional Problems in Rectangular Cartesian Coordinates

For plane elasticity with constant body forces or with body forces derivable from a potential function, the compatibility relations reduce to the following single equation in terms of a stress function F (for simply connected regions):

$$\nabla^2\nabla^2 F = 0 \tag{5-7.1}$$

where for plane rectangular Cartesian coordinates (x, y),

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5-7.2}$$

For zero body force, the stress components σ_x, σ_y, τ_{xy} are related to F by the equations

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y} \tag{5-7.3}$$

In the absence of body forces, Eqs. (5-7.3) automatically satisfy equilibrium [Eqs. (5-2.11)]. Accordingly, any solution to Eq. (5-7.1) represents the solution of a certain problem of plane elasticity. For example, any of the terms of Eq. (5-4.13) represents a solution to Eq. (5-7.1). Hence, Eq. (5-4.13) represents a set of solutions of the problem of plane elasticity.

If the stress function F is taken in the form of a polynomial in x and y, we note [see Eqs. (5-7.3)] that nontrivial (nonzero) stress components are obtained only for a polynomial of second degree or higher in x and y. Furthermore, Eq. (5-7.1) is satisfied identically by polynomials of third degree in x and y. For polynomials of degree higher than three, Eq. (5-7.1) requires the coefficients of all terms of degree higher than three to satisfy a set of $n - 3$ auxiliary conditions, where n is the degree of the polynomial.

For discontinuous loads on boundaries, the polynomial method has severe theoretical limitations, as discontinuous boundary conditions are not representable by polynomials. For continuously varying loads, however, the polynomial method seems to be unlimited theoretically, although in practice the computations may quickly become prohibitive if boundary conditions are to be precisely satisfied. Furthermore, because the computations soon become laborious in any case, the polynomial method requires a systematic approach. One such approach has been proposed by Neou (1957).

Method of Neou. The method proposed by C. Y. Neou (1957) systematically reduces the Airy stress function F expressed in a general doubly infinite power series to the desirable polynomial form for special cases. The method proceeds as follows: Let

$$F = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^m y^n \tag{5-7.4}$$

where $m, n = 0, 1, 2, \ldots$, and A_{mn} are undetermined coefficients that may be arranged in the following rectangular array:

$$\begin{matrix} A_{00} & A_{01} & A_{02} & A_{03} & A_{04} & \cdots \\ A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & \cdots \\ A_{20} & A_{21} & A_{22} & A_{23} & A_{24} & \cdots \\ A_{30} & A_{31} & A_{32} & A_{33} & A_{34} & \cdots \\ A_{40} & A_{41} & A_{42} & A_{43} & A_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix} \tag{5-7.5}$$

Substitution of Eq. (5-7.4) into Eqs. (5-7.3) yields

$$\sigma_x = \sum_{m=0}^{\infty} \sum_{n=2}^{\infty} n(n-1) A_{mn} x^m y^{n-2} \tag{5-7.6}$$

$$\sigma_y = \sum_{m=2}^{\infty} \sum_{n=0}^{\infty} m(m-1) A_{mn} x^{m-2} y^n \tag{5-7.7}$$

$$\tau_{xy} = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn A_{mn} x^{m-1} y^{n-1} \tag{5-7.8}$$

Because A_{00}, A_{01}, and A_{10} do not occur in Eqs. (5-7.6), (5-7.7), and (5-7.8), they may be omitted from Eq. (5-7.5).

Substitution of Eq. (5-7.4) into Eq. (5-7.1) yields

$$\sum_{m=4}^{\infty}\sum_{n=0}^{\infty} m(m-1)(m-2)(m-3)x^{m-4}y^{n}A_{mn} + 2\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} m(m-1)n(n-1)x^{m-2}y^{n-2}A_{mn} + \sum_{m=0}^{\infty}\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)x^{m}y^{n-4}A_{mn} = 0 \qquad (5\text{-}7.9)$$

Collecting similar powers of x and y and writing Eq. (5-7.9) under one summation sign, we obtain

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} [(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2}]x^{m-2}y^{n-2} = 0 \quad (5\text{-}7.10)$$

Because Eq. (5-7.10) must be satisfied for all values of x and y,

$$(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2} = 0 \quad (5\text{-}7.11)$$

Equation (5-7.11) establishes an interrelation among any three alternate coefficients in the diagonals of Eq. (5-7.5), running from lower left to upper right. For example, for $m = 2$ and $n = 2$, Eq. (5-7.11) yields

$$3A_{40} + A_{22} + 3A_{04} = 0$$

Similarly, other relations between the A_{mn} may be established by Eq. (5-7.11).

In the manner outlined above, the plane problem of elasticity with continuous boundary stress is reduced to the determination of A_{mn} [see Eqs. (5-7.4) and (5-7.5)] from the interdependence relations [Eq. (5-7.11)] and the prescribed boundary conditions.

Alternatively, the plane problem of elasticity may be solved by more general techniques, such as transform methods (Milne–Thompson, 1942; Stevenson, 1943; Green, 1945; Sneddon, 1995) or by methods of complex variables (Muskhelishvili, 1975).

Example 5-7.1. Stress Function Compatibility and Stresses. A prismatic cantilever beam has a length L, a rectangular cross section of unit thickness, and a depth $2c$. At its unsupported (free) end it is subjected to an axial tensile load P_1 applied at the centroid of the cross section and a vertical load P_2 parallel to the depth dimension $2c$. By the method of Neou, an engineer develops the following formula for the corresponding Airy stress function:

$$F = \frac{1}{4c}(3P_2xy - P_2xy^3/c^2 + P_1y^2) \tag{a}$$

where x, y are coordinates along the beam and along the depth direction, respectively, with origin at the centroid of the cross section of the free end. We wish to verify the correctness of Eq. (a).

To check the compatibility, we must ensure that F given by Eq. (a) satisfies Eq. (5-7.1). Substitution of Eq. (a) into Eq. (5-7.1) verifies the result $\nabla^2\nabla^2F = 0$. Thus, F is a valid stress function. Next, we must examine the boundary conditions at the unsupported (free) end $x = 0$. By Eqs. (5-7.3) we find

$$\begin{aligned} \sigma_x &= \frac{\partial^2 F}{\partial y^2} = \frac{P_1}{2c} - \frac{3P_2xy}{2c^3} \\ \sigma_y &= \frac{\partial^2 F}{\partial x^2} = 0 \\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x\, \partial y} = -\frac{3P_2(c^2 - y^2)}{4c^3} \end{aligned} \tag{b}$$

At the free end $x = 0$, the stress components must satisfy the conditions

$$\begin{aligned} \int_{-c}^{c} \sigma_x\, dy &= P_1 \\ \int_{-c}^{c} \tau_{xy}\, dy &= -P_2 \end{aligned} \tag{c}$$

Substitution of Eqs. (b) into Eq. (c) verifies that Eqs. (c) are satisfied. At the supported end of the beam, the support at $x = L$ must exert stress components σ_x, τ_{xy} on the beam, as given by Eq. (b), for the solution to be valid throughout the beam.

Problem Set 5-7

1. Determine the interrelations of A_{mn} [Eq. (5-7.11)] for $(m = 4, n = 2)$, $(m = 3, n = 3)$ and $(m = 2, n = 4)$.

2. By the method of Neou, derive a polynomial in x and y for the Airy stress function F for the cantilever beam loaded as shown in Fig. P5-7.2. Hence, derive formulas for the stress components σ_x, σ_y, τ_{xy}. What stress boundary conditions exist at $x = L$? Discuss the application of Saint-Venant's principle to this problem (see Section 4-15 in Chapter 4).

3. By the method of Neou, derive a polynomial in x and y for the Airy stress function F for the beam loaded as shown in Fig. P5-7.3. Hence, derive formulas for the stress components σ_x, σ_y, τ_{xy}. Discuss the application of Saint-Venant's principle to this problem (see Section 4-15).

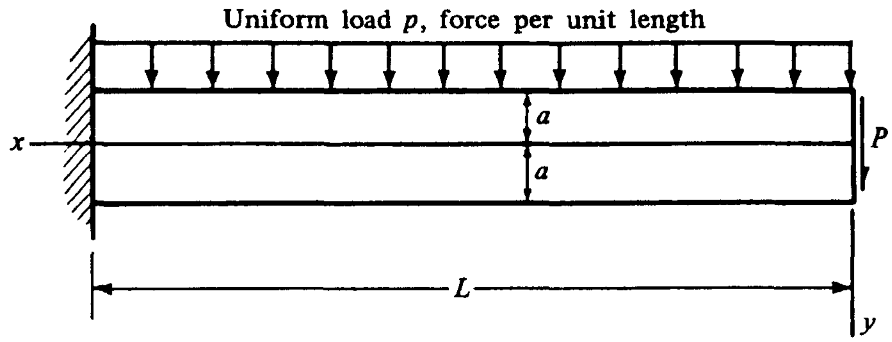


Figure P5-7.2

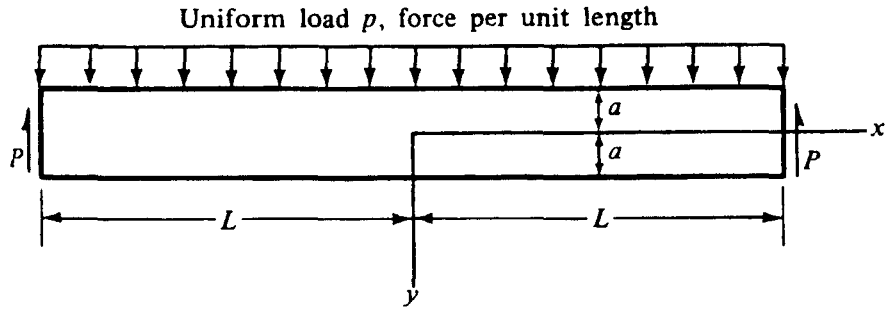


Figure P5-7.3

4. A cantilever beam is loaded as shown in Fig. P5-7.4.
   (a) Derive expressions for the stresses in the beam using the stress function

$$\phi = C_1 xy + C_2 \frac{x^3}{6} + C_3 \frac{x^3 y}{6} + C_4 \frac{xy^3}{6} + C_5 \frac{x^3 y^3}{9} + C_6 \frac{xy^5}{20}$$

At the boundary $x = 0$ the solution is to satisfy the condition that the resultant force system vanishes (that is, $F_x = F_y = M_z = 0$). What stress boundary conditions exist at $x = L$?

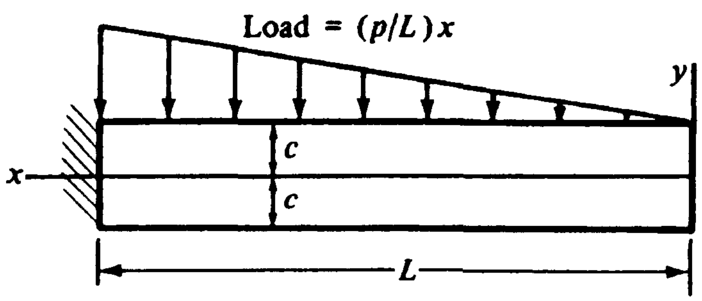


Figure P5-7.4

(b) Derive expressions for the displacement components u and v, assuming that the beam is in a state of plane stress and that it is fixed at the left end so that

$$u(L, 0) = v(L, 0) = 0, \qquad \frac{\partial u}{\partial y}(L, 0) = 0$$

5. The Airy function $F = Ax^3y$ generates a solution for a plane-strain problem with zero body forces. Is this an exact three-dimensional solution? Explain. Determine the stresses and displacements by any valid procedure (Section 5-6).

6. A long prismatic dam is subjected to water pressure that increases linearly with depth. The dam has thickness $2b$ and height h (Fig. P5-7.6). Formulate the stress-determination problem as a well-posed plane problem. State whether the problem is plain strain or generalized plane stress. Relax the boundary conditions at $x = 0$ and $x = h$ to require only restrictions on the *resultant* force system. Solve the problem using the stress function

$$F = A_1xy + A_2x^3 + A_3x^3y + A_4xy^3 + A_5(5x^3y^3 - 3xy^5)$$

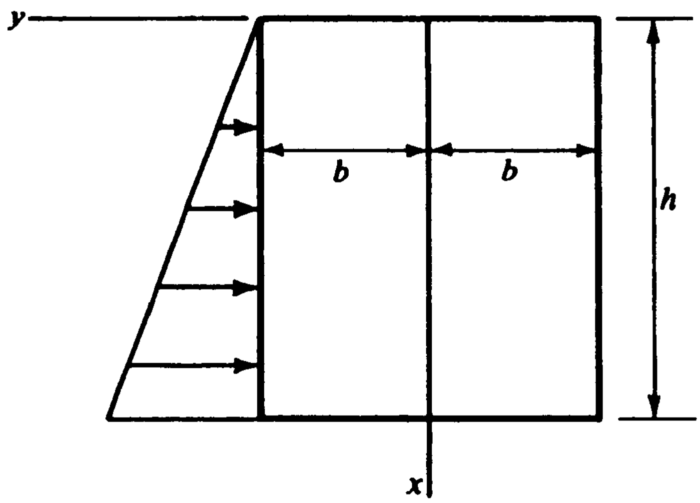


Figure P5-7.6

7. By the method of Neou, the Airy stress function

$$F = \frac{p}{60a}\left(5\,\frac{L^2}{a^2} - 3\right)y^3 + \frac{p}{40a^3}y^5 - \frac{pa}{40L}xy + \frac{p}{20aL}xy^3$$
$$- \frac{p}{40a^3L}xy^5 - \frac{p}{4}x^2 + \frac{3p}{8a}x^2y - \frac{p}{8a^3}x^2y^3 + \frac{p}{12L}x^3$$
$$- \frac{p}{8aL}x^3y + \frac{p}{24a^3L}x^3y^3$$

is obtained for a rectangular beam supported by end shear load and subjected to a triangular load as shown in Fig. P5-7.7. Discuss the validity of the solution.

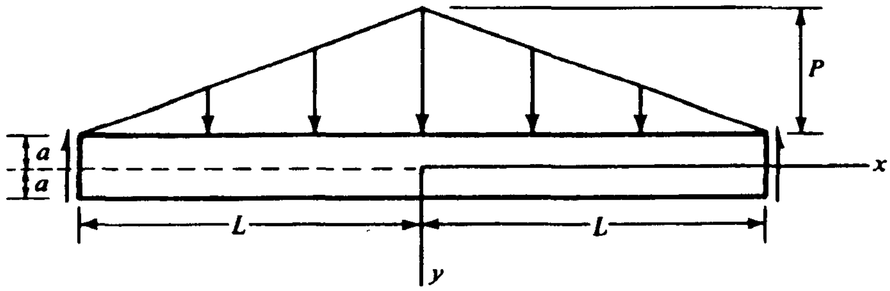


Figure P5-7.7

8. The cantilever beam shown in Fig. P5-7.8 is subjected to a distributed shear stress on the upper face. Assume the stress function for the problem to be of the form

$$F = C_1 y^2 + C_2 y^3 + C_3 y^4 + C_4 y^5 + C_5 x^2 + C_6 x^2 y + C_7 x^2 y^2 + C_8 x^2 y^3$$

The boundary conditions are

$$\text{at} \quad y = -h, \qquad \tau_{xy} = \sigma_y = 0$$
$$\text{at} \quad y = +h, \qquad \tau_{xy} = \frac{-\tau_0 x}{l}, \qquad \sigma_y = 0$$

At the free end, the resultant forces and moment are zero. Determine the eight constants $C_1, C_2, \ldots, C_8$.

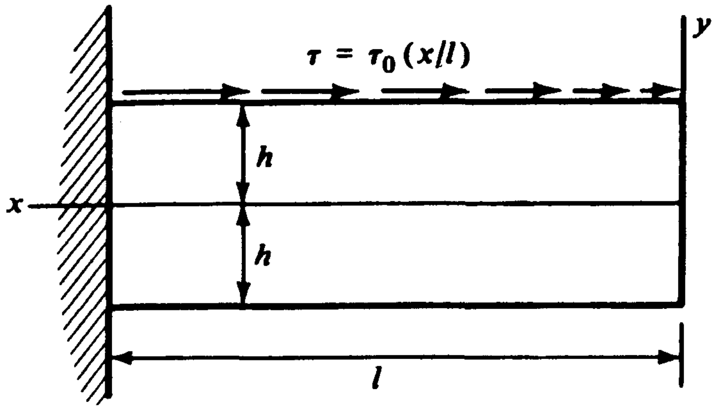


Figure P5-7.8

9. Consider the polynomial $F(x, y) = C_1 x^5 + C_2 x^4 y + C_3 x^3 y^2 + C_4 x^2 y^3 + C_5 x y^4 + C_6 y^5$, where (x, y) are plane rectangular Cartesian coordinates and $C_1, C_2, \ldots, C_6$ are constants.
   (a) Determine the conditions for which $F(x, y)$ is an Airy stress function (that is, for which F is biharmonic).

(b) Derive formulas for the corresponding stress components. Are they compatible?

(c) Let $C_1 = C_3 = C_4 = C_6 = 1$. Specialize the stress formulas accordingly.

(d) Determine the boundary value stress problem for which $F(x, y)$ represents a solution for an isotropic homogeneous elastic medium in the region R bounded by $0 \leq x \leq 1$, $0 \leq y \leq 1$; that is, determine the boundary stresses that act on the region R.

10. The Airy stress function,

$$F = Ax^2 + Bx^2y + Cy^3 + Dy^5 + Ex^2y^3 \tag{a}$$

where $A, B, \ldots, E$ are constants, can be used to get an approximate plane stress solution for a cantilever beam of unit width, length L, and depth $2c$, subject to a uniform pressure q (force/length) on its upper surface. The coordinates (x, y) have origin on the unsupported (free) end at the centroid of the end cross section, with x directed along the axis of the beam and y directed upward.

(a) Determine the requirements on $A, B, \ldots, E$ so that $F(x, y)$ is biharmonic.

(b) Determine the constants $A, B, \ldots, E$ so that the boundary conditions of the problem are satisfied (pressure q for $y = c$; zero net force and net moment on the free end $x = 0$).

5-8 Plane Elasticity in Terms of Displacement Components

In many problems it is convenient to seek solutions in terms of the displacement components. Accordingly, in this section we present equations of plane elasticity relative to the (x, y) plane in terms of (x, y) displacement components (u, v). We consider the case of plane stress, the results for plane strain being obtained in an analogous manner. We employ the approximations of small displacements.

In terms of (x, y) Cartesian coordinates, the strain components ϵ_x, ϵ_y, τ_{xy} in terms of (x, y) displacement components (u, v) are

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5-8.1}$$

Hence, substitution of Eqs. (5-8.1) into Eqs. (5-3.10) yields the stress-displacement relations

$$\begin{aligned} \sigma_x &= \frac{E}{1 - \nu^2}\left[\frac{\partial u}{\partial x} + \nu\,\frac{\partial v}{\partial y} - (1 + \nu)kT\right] \\ \sigma_y &= \frac{E}{1 - \nu^2}\left[\nu\,\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - (1 + \nu)kT\right] \\ \tau_{xy} &= \frac{E}{2(1 + \nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned} \tag{5-8.2}$$

Equations (5-2.11) and (5-8.2) yield (in the absence of body forces and for variable modulus of elasticity E)

$$\begin{aligned}
&u_{xx} + \tfrac{1}{2}(1-\nu)u_{yy} + \tfrac{1}{2}(1+\nu)v_{xy} + (u_x + \nu v_y)\frac{1}{E}\frac{\partial E}{\partial x} \\
&\quad + \tfrac{1}{2}(1-\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial x} \\
&\tfrac{1}{2}(1+\nu)u_{xy} + \tfrac{1}{2}(1-\nu)v_{xx} + v_{yy} + \tfrac{1}{2}(1-\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial x} \\
&\quad + (v_y + \nu u_x)\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial y}
\end{aligned} \tag{5-8.3}$$

where subscripts (x, y) on (u, v) denote partial derivatives. For $E = \text{constant}$, $\partial E/\partial x = \partial E/\partial y = 0$.

Similarly, Eqs. (5-8.1), (5-3.8), and (5-2.11) yield for plane strain

$$\begin{aligned}
&(1-\nu)u_{xx} + \tfrac{1}{2}(1-2\nu)u_{yy} + \tfrac{1}{2}v_{xy} + [(1-\nu)u_x + \nu v_y]\frac{1}{E}\frac{\partial E}{\partial x} \\
&\quad + \tfrac{1}{2}(1-2\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial x} \\
&\tfrac{1}{2}u_{xy} + \tfrac{1}{2}(1-2\nu)v_{xx} + (1-\nu)v_{yy} + \tfrac{1}{2}(1-2\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial x} \\
&\quad + [\nu u_x + (1-\nu)v_y]\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial y}
\end{aligned} \tag{5-8.4}$$

The solution to Eqs. (5-8.3) or (5-8.4) subject to appropriate boundary conditions constitutes the solution of the plane problem of elasticity. Ordinarily, exact solutions to these equations are not readily achieved. Then we may resort to approximate numerical methods. For certain problems the concept of a displacement potential function may be useful (see Example 5-4.1).

Problem Set 5-8

1. Consider the small-displacement plane elasticity problem of plane stress relative to the (x, y) plane. Express the equilibrium equations in terms of (u, v), the (x, y) displacement components, including the effects of temperature $T(x, y)$, and letting the modulus of elasticity E be dependent on (x, y). Include body forces.

2. A state of plane strain relative to the (x, y) plane is defined by $u = u(x, y)$, $v = v(x, y)$, $w = 0$. The strain energy density U_0 of a certain crystal undergoing plane strain is given by

$$U_0 = \tfrac{1}{2}(b_{11}\epsilon_x^2 + b_{22}\epsilon_y^2 + b_{33}\gamma_{xy}^2 + 2b_{12}\epsilon_x\epsilon_y + 2b_{13}\epsilon_x\gamma_{xy} + 2b_{23}\epsilon_y\gamma_{xy})$$

where b_{ij}, $i, j = 1, 2, 3$ are elastic coefficients. For small-displacement theory, derive the differential equations of equilibrium in terms of (u, v) for plane strain of the crystal, including the effects of body forces.

5-9 Plane Elasticity Relative to Oblique Coordinate Axes

In certain classes of plane problems, it is convenient to employ elasticity equations relative to oblique coordinate axes. Accordingly, consider oblique coordinates (ξ, η) with axis ξ coincident with axis x of rectangular Cartesian axes (x, y) and axis η forming angle θ relative to axes ξ (Fig. 5-9.1). Hence, a typical point P in a region may be located by the coordinates (ξ, η) or (x, y), where

$$x = \xi + \eta \cos\theta, \qquad y = \eta \sin\theta \tag{5-9.1}$$

or

$$\xi = x - y \cot\theta, \qquad \eta = y \csc\theta \tag{5-9.2}$$

Under a deformation, the point P goes into the point P^* under displacement components (u, v) relative to axes (x, y) or (U, V) relative to axes (ξ, η), where

$$u = U + V \cos\theta, \qquad v = V \sin\theta \tag{5-9.3}$$

or

$$U = u - v \cot\theta, \qquad V = v \csc\theta \tag{5-9.4}$$

We consider $u = u(x, y)$, $v = v(x, y)$ and $U = U(\xi, \eta)$, $V = V(\xi, \eta)$.

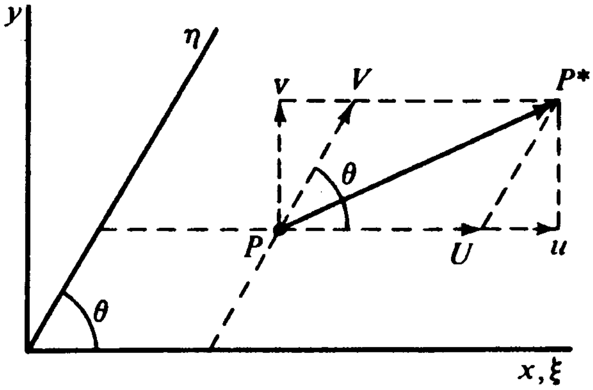


Figure 5-9.1

For small-displacement theory, we obtain from Eqs. (2B-13) in Chapter 2, discarding quadratic terms (and letting $x_1 = x$, $x_2 = y$, $y_1 = \xi$, $y_2 = \eta$, etc.) the strain components $(\epsilon_\xi, \epsilon_\eta, \gamma_{\xi\eta})$ relative to axes (ξ, η) as

$$\begin{aligned} \epsilon_\xi &= \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \xi}\cos\theta \\ \xi_\eta &= \frac{\partial V}{\partial \eta} + \frac{\partial U}{\partial \eta}\cos\theta \\ \gamma_{\xi\eta} &= \frac{\partial V}{\partial \xi} + \frac{\partial U}{\partial \eta} + \left(\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta}\right)\cos\theta \end{aligned} \tag{5-9.5}$$

Also, by the chain rule of partial differentiation and Eqs. (5-9.3), we have for the strain components $(\epsilon_x, \epsilon_y, \gamma_{xy})$

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \xi}\cos\theta \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \xi}\cos\theta \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial U}{\partial \eta} - \frac{\partial V}{\partial \xi}\cos 2\theta\right)\csc\theta + \left(\frac{\partial V}{\partial \eta} - \frac{\partial U}{\partial \xi}\right)\cot\theta \end{aligned} \tag{5-9.6}$$

For $\theta = \pi/2$, Eqs. (5-9.5) and (5-9.6) reduce to the usual results for orthogonal axes. By Eqs. (5-9.5) and (5-9.6), we obtain

$$\begin{aligned} \epsilon_x &= \epsilon_\xi \\ \epsilon_y &= \epsilon_\xi \cot^2\theta + \epsilon_\eta \csc^2\theta - \gamma_{\xi\eta}\cot\theta\csc\theta \\ \gamma_{xy} &= \gamma_{\xi\eta}\csc\theta - 2\epsilon_\xi\cot\theta \end{aligned} \tag{5-9.7}$$

We define stress components $(\sigma_\xi, \sigma_\eta, \tau_{\xi\eta}, \tau_{\eta\xi})$ relative to axes (ξ, η) by considering an element with sides coincident with (ξ, η) coordinates lines (Fig. 5-9.2; see also Problem 3-8.4 in Chapter 3). Hence, considering equilibrium of forces and moments as for the rectangular Cartesian element, we obtain the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_\xi}{\partial \xi} + \frac{\partial \tau_{\eta\xi}}{\partial \eta} + \left(\frac{\partial \tau_{\eta\xi}}{\partial \xi} + \frac{\partial \sigma_\eta}{\partial \eta}\right)\cos\theta &= 0 \\ \frac{\partial \tau_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma_\eta}{\partial \eta} &= 0 \\ \tau_{\xi\eta} &= \tau_{\eta\xi} \end{aligned} \tag{5-9.8}$$

For $\theta = \pi/2$, Eqs. (5-9.8) reduce to the usual equation of equilibrium relative to orthogonal plane axes (x, y).

Relations between stress components $(\sigma_x, \sigma_y, \tau_{xy})$ defined relative to axes (x, y) and $(\sigma_\xi, \sigma_\eta, \tau_{\xi\eta}, \tau_{\eta\xi})$ defined relative to axes (ξ, η) may be derived by considering the

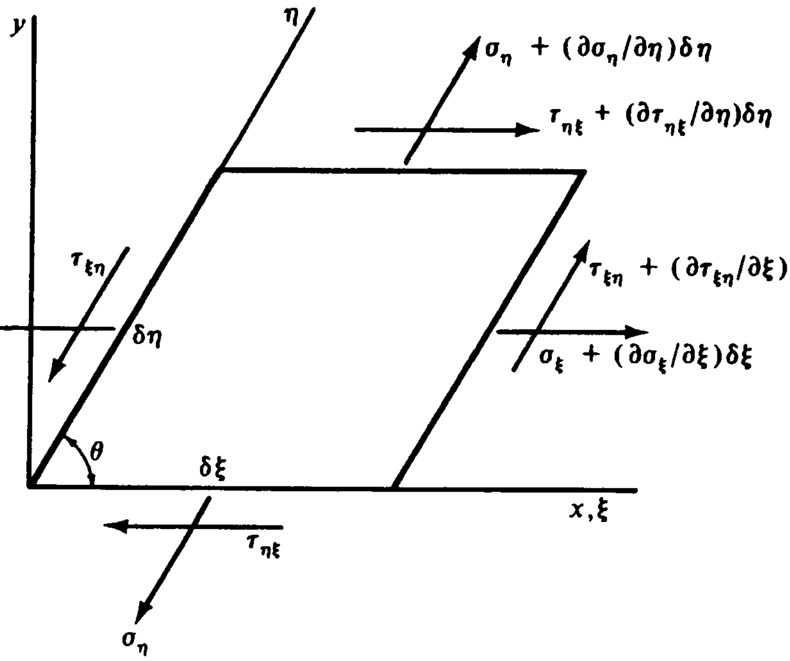


Figure 5-9.2

equilibrium of appropriate elements. Accordingly, by the equilibrium conditions for the elements shown in Fig. 5-9.3, we obtain

$$\begin{aligned} \sigma_\xi &= \sigma_x \sin\theta - 2\tau_{xy}\cos\theta + \sigma_y \cos\theta \cot\theta \\ \sigma_\eta &= \sigma_y \csc\theta \\ \tau_{\xi\eta} &= \tau_{\eta\xi} = \tau_{xy} - \sigma_y \cot\theta \end{aligned} \tag{5-9.9}$$

Substitution of Eqs. (5-9.7) into Eqs. (5-3.8) yields for *plane strain*

$$\begin{aligned} \sigma_x &= K_1[(1 - v + v\cot^2\theta)\epsilon_\xi + v\epsilon_\eta \csc^2\theta - v\gamma_{\xi\eta}\csc\theta\cot\theta - (1+v)kT] \\ \sigma_y &= K_1[(v + \cot^2\theta - v\cot^2\theta)\epsilon_\xi + (1-v)\epsilon_\eta\csc^2\theta \\ &\quad - (1-v)\gamma_{\xi\eta}\csc\theta\cot\theta - (1+v)kT] \\ \tau_{xy} &= K_1\left[\frac{1-2v}{2}\gamma_{\xi\eta}\csc\theta - (1-2v)\epsilon_\xi\cot\theta\right] \end{aligned} \tag{5-9.10}$$

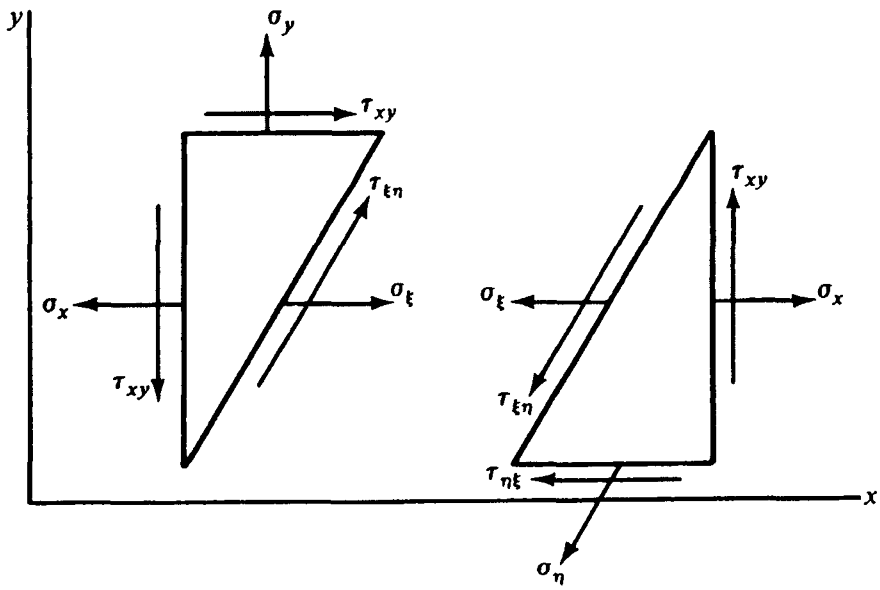


Figure 5-9.3

where

$$K_1 = \frac{E}{(1+v)(1-2v)} \tag{5-9.11}$$

Hence, substitution of Eqs. (5-9.10) into Eqs. (5-9.9) yields the stress–strain relations for plane strain states relative to coordinate axes (ξ, η). Thus, we find

$$\begin{aligned}
\sigma_\xi \sin^3\theta &= K_1[(1-v)\epsilon_\xi + (\cos^2\theta - v\cos 2\theta)\epsilon_\eta \\
&\quad - (1-v)\gamma_{\xi\eta}\cos\theta - (1+v)kT\sin^2\theta] \\
\sigma_\eta \sin^3\theta &= K_1[(\cos^2\theta - v\cos 2\theta)\epsilon_\xi + (1-v)\epsilon_\eta \\
&\quad - (1-v)\gamma_{\xi\eta}\cos\theta - (1+v)kT\sin^2\theta] \\
\tau_{\xi\eta}\sin^3\theta &= K_1[-(1-v)(\epsilon_\xi + \epsilon_\eta) + \tfrac{1}{2}(1-2v+\cos^2\theta)\gamma_{\xi\eta} \\
&\quad + (1+v)kT\sin^2\theta\cos\theta]
\end{aligned} \tag{5-9.12}$$

Similarly, for plane stress [Eqs. (5-3.10)] we obtain

$$\begin{aligned}\sigma_\xi \sin^3\theta &= K_2[\epsilon_\xi + (\cos^2\theta + \nu\sin^2\theta)\epsilon_\eta - \gamma_{\xi\eta}\cos\theta \\ &\quad - (1+\nu)kT\sin^2\theta] \\ \sigma_\eta \sin^3\theta &= K_2[(\cos^2\theta + \nu\sin^2\theta)\epsilon_\xi + \epsilon_\eta - \gamma_{\xi\eta}\cos\theta \\ &\quad - (1+\nu)kT\sin^2\theta] \\ \tau_{\xi\eta}\sin^3\theta &= K_2[-(\epsilon_\xi + \epsilon_\eta)\cos\theta + \tfrac{1}{2}(1 + \cos^2\theta - \nu\sin^2\theta)\gamma_{\xi\eta} \\ &\quad + (1+\nu)kT\sin^2\theta\cos\theta]\end{aligned} \tag{5-9.13}$$

where

$$K_2 = \frac{E}{1-\nu^2} \tag{5-9.14}$$

The preceding equations find application in cases where orthogonal plane axes do not coincide with the boundary curves of the region, for example, in parallelogram regions such as swept-back airplane wings (Fig. 5-9.2). A general development for the theory of shells in nonorthogonal coordinates has been presented by Langhaar (1961).

APPENDIX 5A PLANE ELASTICITY WITH COUPLE STRESSES

5A-1 Introduction

The basic distinction between the classical theory of stress and the theory of stress including couple stresses lies in the nature of the *assumed* interaction of the material on two sides of a surface element. In the classical theory, it is assumed that the action of the material on one side of the surface upon the material on the other side of the surface is equipollent to a force (see Section 3-1 and Fig. 3-1.1 in Chapter 3). In couple-stress theory, the interaction is assumed to be equipollent to a force and a couple (stress couple). Further refinement is also admitted in the nature of assumed body couples (analogous to body forces; see Section 3-8). The couple stresses are taken to be moments per unit area, and the body couples are moments per unit volume.

It has been noted that relatively few practical applications of couple-stress (body-couple) theories are known (Schijve, 1966; Ellis and Smith, 1967; Koiter, 1968). Nevertheless, the theory is less restrictive than the classical stress theory of Euler and Cauchy. Furthermore, applications of the simplest theory of elasticity, in which couple stresses are admitted, to problems in which the analogous classical solutions yield locally unbounded stresses or deformations indicate that the results (for instance, singularities) are changed, softened, or perhaps eliminated (Sternberg, 1968).

Accordingly, in this appendix we give a brief discussion of the linear couple-stress theory for the equilibrium of homogeneous isotropic elastic solids under the conditions of *plane strain*. In particular, we follow the heuristic procedure employed by Mindlin (Mindlin, 1963; Weitsman, 1965; Kaloni and Ariman, 1967). Finally, although the whole of the classical theory of elasticity seems in agreement with the assumption that couple stresses vanish, a study of the couple-stress theory may lead to a critical reexamination of the basic concepts and principles of the mechanics of continuum. In this last regard, one may read with profit the paper by Toupin (1964).

5A-2 Equations of Equilibrium

For the plane problem relative to the (x, y) plane and in the absence of body forces and couples, the stress equations of equilibrium for a medium that can support couple stresses are, in (x, y) notation (see Fig. 5A-2.1; see also Appendix 3B in Chapter 3),

$$
\begin{aligned}
\sum F_x &= 0\colon \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \\
\sum F_y &= 0\colon \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \\
\sum M_0 &= 0\colon \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \tau_{xy} - \tau_{yx} = 0
\end{aligned}
\tag{5A-2.1}
$$

Accordingly, for nonconstant couple stresses ($\partial m_{xz}/\partial x \neq 0$, $\partial m_{yz}/\partial y \neq 0$), the shear stresses are not necessarily equal (that is, $\tau_{xy} \neq \tau_{yx}$). Conversely, if (τ_{xy}, τ_{yx}) are equal to zero, the couple stresses (m_{xz}, m_{yz}) need not vanish. Equations (5A-2.1) are the Cosserat equations of equilibrium for plane problems with body forces and couples omitted (Cosserat and Cosserat, 1909).

5A-3 Deformation in Couple-Stress Theory

We now treat the case of plane strain. As noted in Section 5-1, for plane strain relative to the (x, y) plane, the displacement components (u, v) are functions of (x, y) only and $w = 0$. Hence, for an isotropic elastic medium, the normal strains (ϵ_x, ϵ_y) are related to the normal stresses (σ_x, σ_y) by the first two of Eqs. (5-1.7), and (ϵ_x, ϵ_y) are related to (u, v) by the first two of Eqs. (5-1.4). Furthermore, the shear strain γ_{xy} is related to (u, v) by the fourth of Eqs. (5-1.4). However, because in general $\tau_{xy} \neq \tau_{yx}$, the third of Eqs. (5-1.7) is no longer valid. Hence, following Mindlin (1963), we resolve τ_{xy} and τ_{yx} into a symmetric part τ_S and an antisymmetric part τ_A (see Section 1-25 in Chapter 1 and Fig. 5A-3.1):

$$
\tau_S = \tfrac{1}{2}(\tau_{xy} + \tau_{yz}), \qquad \tau_A = \tfrac{1}{2}(\tau_{xy} - \tau_{yx}) \tag{5A-3.1}
$$

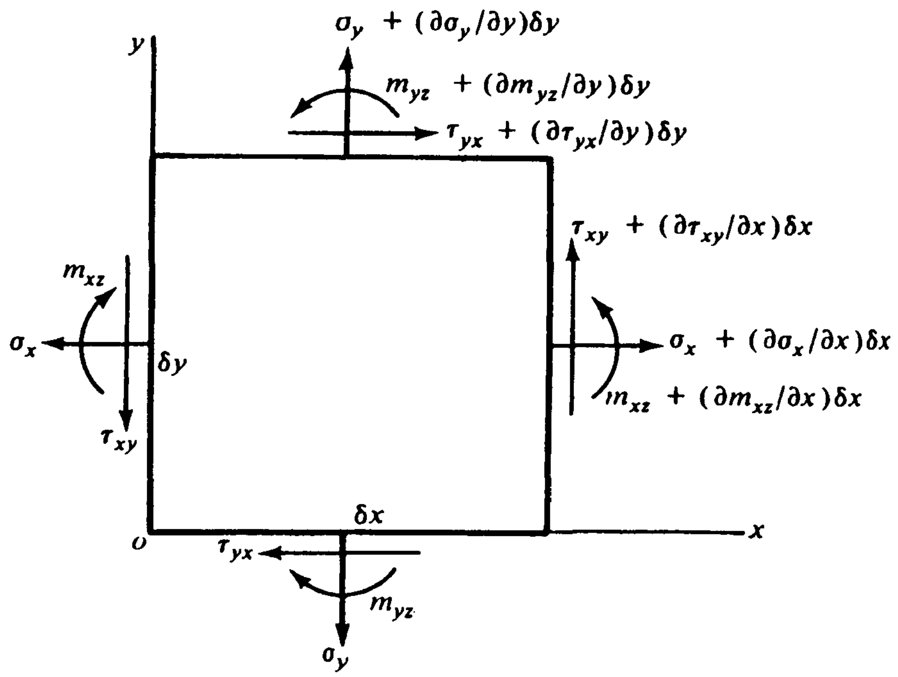


Figure 5A-2.1

Accordingly, by Fig. 5A-3.1 and Section 2-8 in Chapter 2, the symmetric part τ_S produces the shear strain

$$\gamma_{xy} = \frac{1}{G}\tau_S = \frac{1+v}{E}(\tau_{xy} + \tau_{yx}) \tag{5A-3.2}$$

where $G = E/[2(1+v)]$ is the modulus of shear. Similarly, the antisymmetric part τ_A produces a local rigid rotation (Fig. 5A-3.1 and Section 2-13)

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{5A-3.3}$$

Furthermore, the antisymmetric part τ_A is balanced by the couple stresses [Eq. (5A-2.1)].

Considering the effect of the couple stresses on the element $(\delta x, \delta y)$, Fig. 5A-3.2, we note that m_{xz}, m_{yz} produce curvatures κ_{xz} and κ_{yz} related to the rotation ω_z by the equations

$$R_{xz}\frac{\partial \omega_z}{\partial x}\delta x = \delta x, \qquad R_{yz}\frac{\partial \omega_z}{\partial y}\delta y = \delta y$$

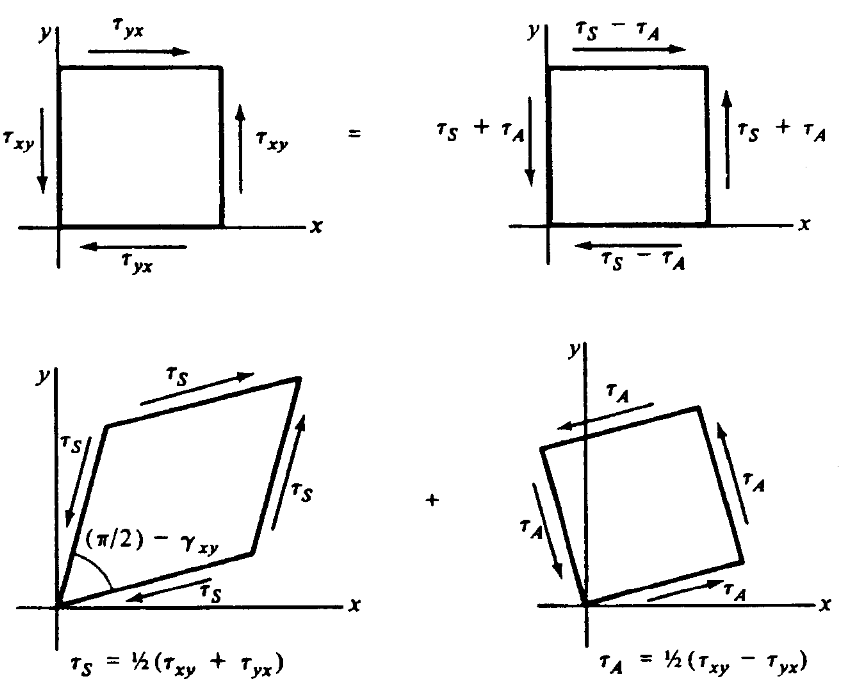


Figure 5A-3.1

or

$$\kappa_{xz} = \frac{\partial \omega_z}{\partial x}, \qquad \kappa_{yz} = \frac{\partial \omega_z}{\partial y} \tag{5A-3.4}$$

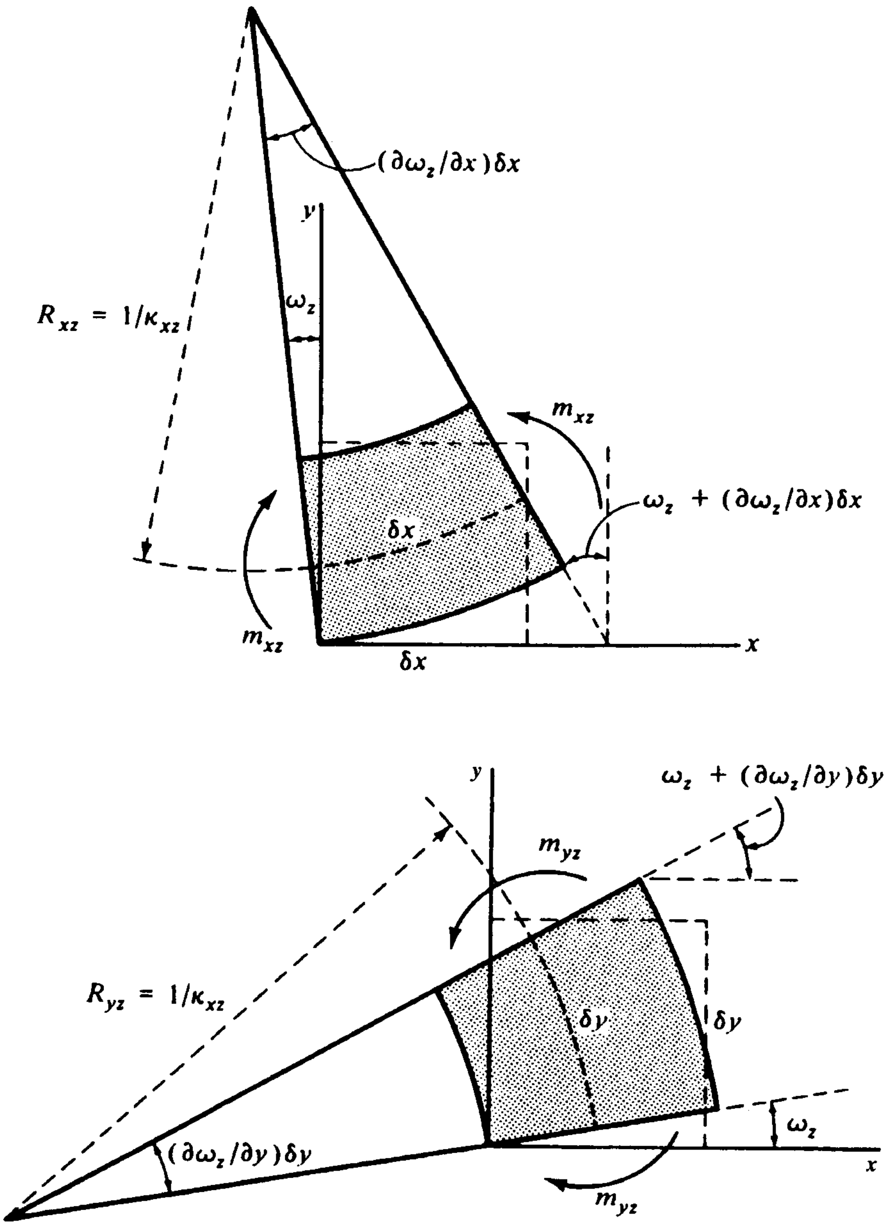
Analogous to the shearing strain γ_{xy} relation to the symmetric part τ_S of τ_{xy}, τ_{yz}, we assume that the curvatures $(\kappa_{xz}, \kappa_{yz})$ (deformations) are proportional to the stress couples (m_{xz}, m_{yz}) (forces):

$$\kappa_{xz} = \frac{1}{4B} m_{xz}, \qquad \kappa_{yz} = \frac{1}{4B} m_{yz} \tag{5A-3.5}$$

where B [see Eq. (5A-3.2)] is a modulus of curvature or bending, and the factor 4 is taken for convenience in later calculations. We note that because the couple stresses have the dimensions of couple per unit area or force per unit length and curvature is the reciprocal of length, the modulus B has the dimensions of force.

5A-4 Equations of Compatibility

Equations (5-1.4), (5A-3.3), and (5A-3.4) consist of five deformation quantities $(\epsilon_x, \epsilon_y, \gamma_{xy}, \kappa_{xz}, \kappa_{yz})$ expressed in terms of two displacement components. By



(∂ωz/∂x)δx
y
Rxz = 1/κxz
ωz
mxz
ωz + (∂ωz/∂x)δx
δx
mxz
δx
x
y
ωz + (∂ωz/∂y)δy
myz
Ryz = 1/κxz
δy
δy
ωz
(∂ωz/∂y)δy
x
myz

Figure 5A-3.2

elimination of the displacement components from Eqs. (5-1.4), we obtain the usual equations of strain compatibility [Eq. (5-3.1)].

Similarly, elimination of the rotation ω_z from Eqs. (5A-3.4) yields

$$\frac{\partial \kappa_{xz}}{\partial y} = \frac{\partial \kappa_{yz}}{\partial x} \tag{5A-4.1}$$

Now, by Eqs. (5-1.4) and (5A-3.3), we find

$$\frac{\partial \omega_z}{\partial x} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x\, \partial y}\right) = \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y}$$

$$\frac{\partial \omega_z}{\partial y} = \frac{1}{2}\left(\frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y}$$

Hence, by Eqs. (5A-3.4),

$$\begin{aligned} \kappa_{xz} &= \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y} \\ \kappa_{yz} &= \frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} \end{aligned} \tag{5A-4.2}$$

Seemingly, we have obtained four compatibility relations [Eqs. (5-3.1), (5A-4.1), and (5A-4.2)]. However, we observe that Eqs. (5-4.2) imply Eq. (5A-4.1). Hence, we have the compatibility relations

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \\ \frac{\partial \kappa_{xz}}{\partial y} &= \frac{\partial \kappa_{yz}}{\partial x} \\ \kappa_{xz} &= \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y} \\ \kappa_{yz} &= \frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} \end{aligned} \tag{5A-4.3}$$

where only three relations are independent, as the second equation is implied by the remaining three.

Finally, we note that the four compatibility relations may be written in terms of stress components ($\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx}$) and couple stresses ($m_{xz}, m_{yz}$) by means of Eqs. (5A-3.2), (5A-3.5), and the first two of Eqs. (5-1.7). Thus, we obtain

$$\begin{aligned} \frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - v\nabla^2(\sigma_x + \sigma_y) &= \frac{\partial^2}{\partial x\, \partial y}(\tau_{xy} + \tau_{yx}) \\ \frac{\partial m_{xz}}{\partial y} &= \frac{\partial m_{yz}}{\partial x} \\ m_{xz} &= l^2 \frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - 2l^2 \frac{\partial}{\partial y}[\sigma_x - v(\sigma_x + \sigma_y)] \\ m_{yz} &= 2l^2 \frac{\partial}{\partial x}[\sigma_y - v(\sigma_x + \sigma_y)] - l^2 \frac{\partial}{\partial y}(\tau_{xy} + \tau_{yx}) \end{aligned} \tag{5A-4.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5A-4.5}$$

and

$$l^2 = \frac{2(1+v)B}{E} = \frac{B}{G} \tag{5A-4.6}$$

where l^2 is the ratio of the material constants, B and G. By the last two of Eqs. (5A-4.4), we note that large stress gradients may lead to large values of the couple stresses (m_{xz}, m_{yz}) when $l^2 \neq 0$. If $l = 0$, the material has relatively no resistance to curvature effects ($B/G = 0$), Eqs. (5A-3.5). Because the second of Eqs. (5A-4.4) is implied by the other three equations, only three of the four compatibility equations are independent.

5A-5 Stress Functions for Plane Problems with Couple Stresses

Equation (5A-2.1) may be solved by means of stress functions in a manner analogous to the solution of Eqs. (5-4.1) by means of the Airy stress function (Carlson, 1966).

According to the theory of total differentials (Section 1-19 in Chapter 1 and Section 5-4), the first of Eqs. (5A-2.1) is a necessary and sufficient condition for the existence of a function ϕ of (x, y) such that

$$\sigma_x = \frac{\partial \phi}{\partial y}, \qquad \tau_{yx} = -\frac{\partial \phi}{\partial x} \tag{5A-5.1}$$

and the second of Eqs. (5A-2.1) yields in a similar manner

$$\sigma_y = \frac{\partial \theta}{\partial x}, \qquad \tau_{xy} = -\frac{\partial \theta}{\partial y} \tag{5A-5.2}$$

where $\theta = \theta(x, y)$. Furthermore, the second of Eqs. (5A-4.4) admits a function $\psi = \psi(x, y)$ such that

$$m_{xz} = \frac{\partial \psi}{\partial x}, \qquad m_{yz} = \frac{\partial \psi}{\partial y} \tag{5A-5.3}$$

Substitution of Eqs. (5A-5.1), (5A-5.2), and (5A-5.3) into the last of Eqs. (5A-2.1) yields

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x} + \phi\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y} - \theta\right) = 0 \tag{5A-5.4}$$

which in turn is a necessary and sufficient condition that the function $H = H(x, y)$ exists, such that

$$\frac{\partial \psi}{\partial x} + \phi = \frac{\partial H}{\partial y}, \qquad \frac{\partial \psi}{\partial y} - \theta = -\frac{\partial H}{\partial x} \tag{5A-5.5}$$

or

$$\phi = \frac{\partial H}{\partial y} - \frac{\partial \psi}{\partial x}, \qquad \theta = \frac{\partial H}{\partial x} + \frac{\partial \psi}{\partial y} \tag{5A-5.6}$$

Hence, substitution of Eqs. (5A-5.6) into Eqs. (5A-5.1) and (5A-5.2) yields expressions for σ_x, σ_y, τ_{yx}, τ_{xy} in terms of ψ and H. Thus, we obtain the formulas

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 H}{\partial y^2} - \frac{\partial^2 \psi}{\partial x\, \partial y}, & \sigma_y &= \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 \psi}{\partial x\, \partial y} \\
\tau_{xy} &= -\frac{\partial^2 H}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y^2}, & \tau_{yx} &= -\frac{\partial^2 H}{\partial x\, \partial y} + \frac{\partial^2 \psi}{\partial x^2} \\
m_{xz} &= \frac{\partial \psi}{\partial x}, & m_{yz} &= \frac{\partial \psi}{\partial y}
\end{aligned} \tag{5A-5.7}$$

where all components of stress and couple stress are expressed in terms of the two stress functions H and ψ. For $\psi = 0$, $m_{xz} = m_{yz} = 0$, and Eqs. (5A-5.7) reduce to the classical Airy stress function relations [Eqs. (5-4.9) with $V = 0$].

Differential Equations for H ***and*** ψ. The remaining equations [Eqs. (5A-4.4)] of compatibility define the functions H and ψ. Hence, substitution of the first four of

Eqs. (5A-5.7) into the first of Eqs. (5A-4.4) yields

$$\nabla^2\nabla^2 H = \nabla^4 H = 0 \tag{5A-5.8}$$

Thus, H is the Airy stress function of classical stress theory [see (Eq. 5-4.12)]. Finally, substitution of Eqs. (5A-5.7) into the last two of Eqs. (5A-4.4) yields

$$\begin{aligned} \frac{\partial}{\partial x}(\psi - l^2\nabla^2\psi) &= -2(1-\nu)l^2 \frac{\partial}{\partial y}(\nabla^2 H) \\ \frac{\partial}{\partial y}(\psi - l^2\nabla^2\psi) &= 2(1-\nu)l^2 \frac{\partial}{\partial x}(\nabla^2 H) \end{aligned} \tag{5A-5.9}$$

Accordingly the functions $\psi - l^2\nabla^2\psi$ and $2(1-\nu)l^2\nabla^2 H$ are conjugate harmonic functions; that is, they satisfy the Cauchy–Riemann equations [see Eqs. (5-5.3)]. By Eqs. (5A-5.9), we obtain, by differentiating the first of Eqs. (5A-5.9) by x and the second by y, and adding,

$$\nabla^2\psi - l^2\nabla^4\psi = 0 \tag{5A-5.10}$$

Similarly, differentiations with respect to y first and x yield Eqs. (5A-5.8). Thus, the defining equations for H and ψ are Eqs. (5A-5.8) and (5A-5.10). The theory of plane strain with couple stresses is contained in Sections 5A-2 through 5A-5. The theory of plane stress may be derived in an analogous manner. In Appendix 6A, the plane strain theory is applied to the problem of a circular hole in a field of uniform tension as well as in a biaxial field of stress.

APPENDIX 5B PLANE THEORY OF ELASTICITY IN TERMS OF COMPLEX VARIABLES

The material treated in Sections 5-5 and 5-6 is essential for the topics discussed in this appendix.

5B-1 Airy Stress Function in Terms of Analytic Functions $\psi(z)$ and $\chi(z)$

It may be shown that the Airy (biharmonic) stress function $F(x, y)$ may be expressed in terms of two analytic functions of the complex variable $z = x + iy$ (Muskhelishvili, 1975). By this result, we transform the plane theory of elasticity into complex variable theory.

In Section 5-5 we introduced the analytic function $\psi(z) = q_1 + iq_2$ and noted that $F - xq_1 - yq_2$ is harmonic, where $i = \sqrt{-1}$, (q_1, q_2) are conjugate harmonic functions and F is the Airy (biharmonic) stress function. Hence, the Airy stress function may be written in the forms [see Eqs. (5-5.11), (5-5.12), and (5-5.13)]:

$$\begin{aligned} F &= xq_1 + yq_2 + h_1 \\ F &= 2xq_1 + h_2 \\ F &= 2yq_2 + h_3 \end{aligned} \tag{5B-1.1}$$

where h_1, h_2, h_3 are arbitrary harmonic functions in the plane region D.

By the appropriate combination of two analytic functions defined in D, we now note that we may generate the Airy stress function in the form of the first of Eqs. (5B-1.1). To do this, we first introduce the analytic function

$$\chi(z) = p_1 + ip_2(z)$$

where (p_1, p_2) are conjugate harmomic functions. Next, we form the real part of $\bar{z}\psi(z) + \chi(z)$, where $\psi(z)$ is defined by Eq. (5-5.6) and $\bar{z} = x - iy$. Thus, we obtain

$$\begin{aligned} \mathrm{Re}[\bar{z}\psi(z) + \chi(z)] &= \mathrm{Re}[(x - iy)(q_1 + iq_2) + p_1 + ip_2] \\ &= xq_1 + yq_2 + p_1 \end{aligned} \tag{5B-1.2}$$

Accordingly, comparison of the first of Eqs. (5B-1.1) and Eq. (5B-1.2) yields

$$F = \mathrm{Re}[\bar{z}\psi(z) + \chi(z)] \tag{5B-1.3}$$

Alternatively, Eq. (5B-1.3) may be written more symmetrically by employing the complex conjugations of ψ and χ and noting that the sum of a complex function and its conjugate yields a real function. Thus,

$$\bar{z}\psi(z) + z\overline{\psi(z)} + \chi(z) + \overline{\chi(z)} = 2(xq_1 + yq_2 + p_1)$$

Hence, we may write F in the form

$$2F = \bar{z}\psi(z) + z\overline{\psi(z)} + \chi(z) + \overline{\chi(z)} \tag{5B-1.4}$$

Equations (5B-1.3) and (5B-1.4) express the Airy stress function F in terms of the two analytic functions $\psi(z)$ and $\chi(z)$ and their complex conjugates. It is readily shown that Eq. (5B-1.4) satisfies the condition $\nabla^2\nabla^2 F = 0$.

5B-2 Displacement Components in Terms of Analytic Functions $\psi(z)$ and $\chi(z)$

For the case of plane stress, the (x, y) displacement components in terms of the Airy stress function F and the complex conjugate harmonic functions (q_1, q_2) are given by Eq. (5-6.8). By Eq. (5B-1.4), we obtain

$$\begin{aligned} \frac{\partial F}{\partial x} &= \tfrac{1}{2}[\psi(z) + \bar{z}\psi'(z) + \overline{\psi(z)} + z\overline{\psi'(z)} + \chi'(z) + \overline{\chi'(z)}] \\ \frac{\partial F}{\partial y} &= \frac{i}{2}[-\psi(z) + \bar{z}\psi'(z) + \overline{\psi(z)} - z\overline{\psi'(z)} + \chi'(z) - \overline{\chi'(z)}] \end{aligned} \tag{5B-2.1}$$

where primes denote differentiation with respect to z.

In developing the theory, it is expedient to express quantities in terms of $\partial F/\partial x + i(\partial F/\partial y)$. Hence, by Eq. (5B-1.2) we write

$$\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y} = \psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)} \tag{5B-2.2}$$

Consequently, multiplication of the second of Eqs. (5-6.8) by i and addition to the first of Eqs. (5-6.8) yields, with Eq. (5B-2.2),

$$2G(u + iv) = \kappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)} \tag{5B-2.3}$$

where for plane stress (also generalized plane stress)

$$\kappa = \frac{3 - \nu}{1 + \nu} \qquad \text{(plane strain)} \tag{5B-2.4}$$

In an analogous manner, we also obtain Eq. (5B-2.3) for the case of plane strain, where for plane strain

$$\kappa = 3 - 4\nu \qquad \text{(plane stress)} \tag{5B-2.5}$$

One may transform the expression for plane strain into the equivalent expression for plane stress by the following substitutions:

$$\begin{aligned} \frac{1 - \nu^2}{E} \quad &\text{(plane strain)} \rightarrow \frac{1}{E} \quad \text{(plane stress)} \\ \nu \quad &\text{(plane strain)} \rightarrow \frac{\nu}{1 + \nu} \quad \text{(plane stress)} \end{aligned} \tag{5B-2.6}$$

Equation (5B-2.3) is the fundamental displacement relation in the complex variable theory of plane elasticity.

5B-3 Stress Components in Terms of $\psi(z)$ and $\chi(z)$

Consider a line element AB joining two points in a medium in the (x, y) plane, with positive direction from A to B. Axes n, t are normal and tangential respectively to AB at point P. They form a right-handed coordinate system as do (x, y) (Fig. 5B-3.1). Let the forces $\sigma_{nx}\,ds$, $\sigma_{ny}\,ds$ act on the infinitesimal element ds, with positive sense in the directions of positive (x, y), respectively. Hence, the stress components acting on an element of the medium with sides dx, dy, ds (Fig. 5B-3.2) are σ_x, σ_y, τ_{xy}, σ_{nx}, σ_{ny}. For plane equilibrium of the element, we have (in the absence of body forces)

$$\begin{aligned} \textstyle\sum F_x &= \sigma_{nx}\,ds - \sigma_x\,dy + \tau_{xy}\,dx = 0 \\ \textstyle\sum F_y &= \sigma_{ny}\,ds + \sigma_y\,dx - \tau_{xy}\,dy = 0 \end{aligned}$$

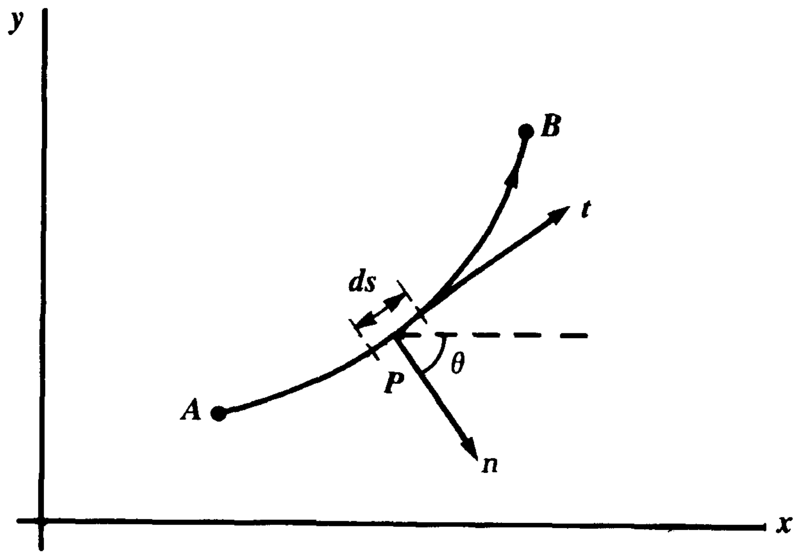


Figure 5B-3.1

or

$$\begin{aligned}\sigma_{nx} &= \sigma_x \cos\theta - \tau_{xy}\sin\theta \\ \sigma_{ny} &= -\sigma_y \sin\theta + \tau_{xy}\cos\theta\end{aligned} \tag{a}$$

where

$$\cos\theta = \frac{dy}{ds}, \qquad \sin\theta = \frac{dx}{ds} \tag{b}$$

Expressing σ_x, σ_y, and τ_{xy} in terms of the stress function F, we may write Eq. (a), with Eqs. (b), in the form

$$\sigma_{nx} = \frac{\partial^2 F}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 F}{\partial x\,\partial y}\frac{dx}{ds} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\frac{dx}{ds} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)\frac{dy}{ds}$$

$$\sigma_{ny} = -\frac{\partial^2 F}{\partial x^2}\frac{dx}{ds} - \frac{\partial^2 F}{\partial x\,\partial y}\frac{dy}{ds} = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)\frac{dx}{ds} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\frac{dy}{ds}$$

Accordingly, we may write by the chain rule of differentiation

$$\sigma_{nx} = \frac{d}{ds}\left(\frac{\partial F}{\partial y}\right), \qquad \sigma_{ny} = -\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) \tag{5B-3.1}$$

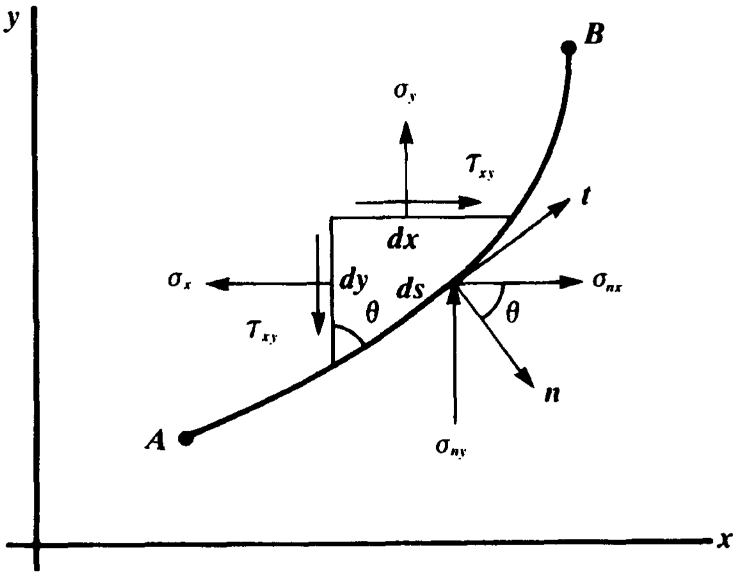


Figure 5B-3.2

Hence, multiplying the second of Eqs. (5B-3.1) by i and adding it to the first of Eqs. (5B-3.1), we obtain

$$\sigma_{nx} + i\sigma_{ny} = \frac{d}{ds}\left(\frac{\partial F}{\partial y} - i\,\frac{\partial F}{\partial x}\right) = -i\,\frac{d}{ds}\left(\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y}\right)$$

or (5B-3.2)

$$(\sigma_{nx} + i\sigma_{ny})\,ds = -id\left(\frac{\partial F}{\partial x} + i\,\frac{\partial F}{\partial y}\right)$$

Substituting Eq. (5B-2.2) into Eq. (5B-3.2), we obtain

$$(\sigma_{nx} + i\sigma_{ny})\,ds = -id[\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}] \tag{5B-3.3}$$

Now let ds have the direction of the y axis. Then $ds = dy$, $dz = i\,dy$, $d\bar{z} = -i\,dy$, $\sigma_{nx} = \sigma_x$, and $\sigma_{ny} = \tau_{xy}$. Then, Eq. (5B-3.3) becomes

$$(\sigma_x + i\tau_{xy}) = \psi'(z) + \overline{\psi'(z)} - z\overline{\psi''(z)} - \overline{\chi''(z)} \tag{5B-3.4}$$

Similarly, let ds have the direction of the x axis. Then $ds = dx$, $dz = dx$, $d\bar{z} = dx$, $\sigma_{nx} = -\tau_{xy}$, and $\sigma_{ny} = -\sigma_y$, and Eq. (5B-3.3) becomes

$$(\sigma_y - i\tau_{xy}) = \psi'(z) + \overline{\psi'(z)} + z\overline{\psi''(z)} + \overline{\chi''(z)} \tag{5B-3.5}$$

Adding and subtracting Eqs. (5B-3.4) and (5B-3.5), we find

$$\begin{aligned}\nabla^2 F = \sigma_x + \sigma_y = 2[\psi'(z) + \overline{\psi'(z)}] = 4\mathrm{Re}[\psi'(z)] \\ \sigma_y - \sigma_x - 2i\tau_{xy} = 2[z\overline{\psi''(z)} + \overline{\chi''(x)}]\end{aligned} \tag{5B-3.6}$$

or by complex conjugation we obtain from the second of Eqs. (5B-3.6)

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\psi''(z) + \chi''(z)] \tag{5B-3.7}$$

where $\psi(z)$ and $\chi(z)$ are analytic functions.

Accordingly, Eqs. (5B-2.3), (5B-3.6), and (5B-3.7) express the components (u, v) of the displacement vector and the components $(\sigma_x, \sigma_y, \tau_{xy})$ of the stress tensor in terms of analytic functions $\psi(z)$ and $\chi(z)$, inside region D occupied by the plane body under consideration.

5B-4 Expressions for Resultant Force and Resultant Moment

Let (F_x, F_y) be the resultant force that acts on an arc AB. Then, by Eqs. (5B-3.2) and (5B-3.3),

$$\begin{aligned}F_x + iF_y &= \int_A^B (\sigma_{nx} + i\sigma_{ny})\, ds \\ &= -i\int_A^B d[\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}] \\ &= -i[\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}]|_A^B\end{aligned} \tag{5B-4.1}$$

Similarly, the moment M with respect to origin 0 of coordinate system (x, y) of the forces that act on AB is (Fig. 5B-3.2)

$$\begin{aligned}M &= \int_{AB} (x\sigma_{ny} - y\sigma_{nx})\, ds \\ &= -\int_{AB}\left[xd\left(\frac{\partial F}{\partial x}\right) + yd\left(\frac{\partial F}{\partial y}\right)\right] \\ &= -\left[x\,\frac{\partial F}{\partial x} + y\,\frac{\partial F}{\partial y}\right]_A^B + \int_{AB}\left[\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds}\right] ds \\ &= -\left[x\,\frac{\partial F}{\partial x} + y\,\frac{\partial F}{\partial y}\right]_A^B + \int_{AB}\frac{dF}{ds}\, ds = -\left[x\,\frac{\partial F}{\partial x} + y\,\frac{\partial F}{\partial y}\right]_A^B + F|_A^B\end{aligned} \tag{5B-4.2}$$

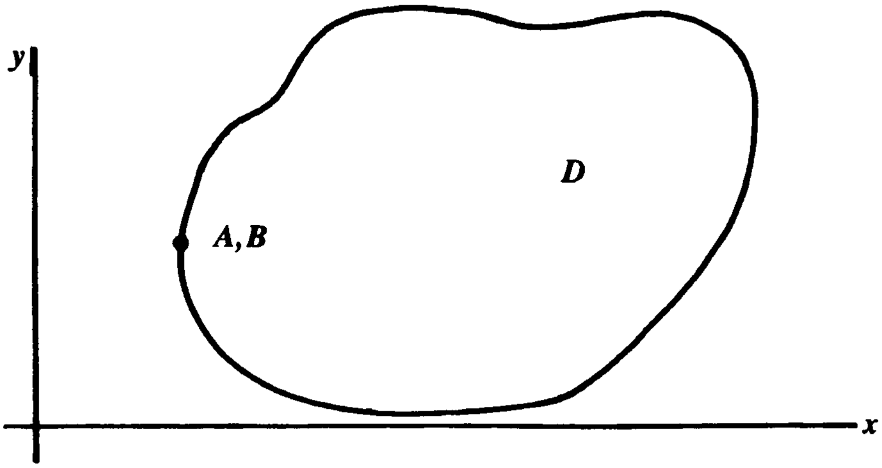


Figure 5B-4.1

Also,

$$x\,\frac{\partial F}{\partial x} + y\,\frac{\partial F}{\partial y} = \mathrm{Re}\left[z\left(\frac{\partial F}{\partial x} - i\,\frac{\partial F}{\partial y}\right)\right] \tag{5B-4.3}$$

Now, by Eq. (5B-2.1), we obtain

$$\frac{\partial F}{\partial x} - i\,\frac{\partial F}{\partial y} = \overline{\psi(z)} + \bar{z}\psi'(z) + \chi'(z) \tag{5B-4.4}$$

Accordingly, by Eqs. (5B-1.3), (5B-4.2), (5B-4.3) and (5B-4.4), the expression for M may be written

$$M = \mathrm{Re}[\chi(z) - z\chi'(z) - z\bar{z}\psi'(z)]_A^B \tag{5B-4.5}$$

Equations (5B-4.1) and (5B-4.5) represent boundary conditions for resultant force and moment in terms of the analytic functions $\psi(z)$ and $\chi(z)$.

Because we have assumed that region D is simply connected, the function $\psi(z)$ and $\chi(z)$ are single-valued. Hence, if points A and B coincide (Fig. 5B-4.1), the curve AB is closed, and the values of ψ and χ are the same at points A and B. Hence, if $A = B$, Eqs. (5B-4.1) and (5B-4.5) yield $F_x = F_y = M = 0$. Thus, for simply connected plane regions, the external forces acting on any part of the region contained inside a closed contour AB is statically equivalent to zero.

5B-5 Mathematical Form of Functions $\psi(z)$ and $\chi(z)$

In this section we consider the degree of arbitrariness of the functions ψ, χ in the cases when (a) the state of stress is given and (b) the displacement field is specified. It is convenient to treat these cases separately. Because $\chi(z)$ occurs in the stress and displacement relations only in the forms $\chi'(z)$ and $\chi''(z)$, it is expedient to define a function $\phi(z)$ such that

$$\chi'(z) = \phi(z) \tag{5B-5.1}$$

Case A. Stress State Given. By Eqs. (5B-5.1), (5B-3.6), and (5B-3.7),

$$\begin{aligned} \sigma_x + \sigma_y &= 2[\psi'(z) + \overline{\psi'(z)}] = 4\text{Re}[\psi'(z)] \\ \sigma_y - \sigma_x - 2i\tau_{xy} &= 2[\bar{z}\psi''(z) + \phi'(z)] \end{aligned} \tag{5B-5.2}$$

To determine the nature of $\psi(z)$, $\phi(z)$, we first note that for the simply connected region D, $\psi(z)$, $\phi(z)$ may be specified to within certain arbitrary complex numbers without altering the stress distribution in region R. Thus, the stress quantities $\sigma_x + \sigma_y$, $\sigma_y - \sigma_x + 2i\tau_{xy}$ may be expressed in terms of either the functions (ψ, ϕ) or the functions (ψ_1, ϕ_1), where

$$\begin{aligned} \psi_1(z) &= \psi(z) + icz + a \\ \phi_1(z) &= \phi(z) + b \end{aligned} \tag{5B-5.3}$$

where (a, b) are complex numbers and c is a real constant. Equations (5B-5.3) follow directly from substitution of $\psi(z)$, $\phi(z)$ and $\psi_1(z)$, $\phi_1(z)$ into Eqs. (5B-5.2) and equating the quantities so obtained (that is, requiring the same stresses for either set of functions). Integration then yields Eqs. (5B-5.3). In other words, if the state of stress in D is specified, the analytic functions ψ, ϕ are determined to within a linear function $icz + a$ and a complex constant b, respectively.

Case B. Displacement Specified. Let us specify the displacement components (u, v) in region D. By Eqs. (5B-5.1) and (5B-2.3), we find

$$\begin{aligned} 2G(u + iv) &= \kappa\psi(z) - z\overline{\psi'(z)} - \overline{\phi(z)} \\ \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= \frac{(1 + \nu)(1 + \kappa)}{E} \text{ Im } \psi'(z) \end{aligned} \tag{5B-5.4}$$

where ω is the volumetric rotation and Im $\psi'(z)$ denotes the imaginary value of $\psi'(z)$, that is,

$$\text{Im } \psi'(z) = -\frac{i}{2}[\psi'(z) - \overline{\psi'(z)}] \tag{5B-5.5}$$

The second of Eqs. (5B-5.4) follows from the fact that by the first of Eqs. (5B-5.4)

$$\begin{aligned} 4Gu &= \kappa[\psi(z) + \overline{\psi(z)}] - z\overline{\psi'(z)} - \bar{z}\psi'(z) - \phi(z) - \overline{\phi(z)} \\ 4Gv &= -i\kappa[\psi(z) - \overline{\psi(z)}] + i[z\overline{\psi'(z)} - \bar{z}\psi'(z) - \phi(z) + \overline{\phi(z)}] \end{aligned} \tag{5B-5.6}$$

Because the stresses are determined uniquely, when the displacements are given, we conclude that the extent of the arbitrariness in the functions ψ, ϕ can be no greater than that exhibited by Eqs. (5B-5.3). Indeed, the requirement that the functions (ψ, ϕ) and (ψ_1, ϕ_1) yield the same displacements demands that

$$c = 0, \qquad \kappa a = \bar{b} \tag{5B-5.7}$$

This restriction is more severe than that of Eq. (5B-5.3). Thus, if the displacements (u, v) are prescribed in D, the function $\psi(z)$ is determined to within a complex constant a and the specification of a defines the constant b. Accordingly, the functions $\psi(z)$, $\phi(z)$ are determined uniquely for a given state of stress, provided a, b, c are chosen so that for the plane region D, the displacement and rotation are specified to account for rigid-body motion. For example, we may specify the displacement and rotation for some point—say, z_0—in R. Then, for example, the conditions

$$\psi(z_0) = 0, \qquad \text{Im}\ \psi'(z_0) = 0, \qquad \phi(z_0) = 0 \tag{5B-5.8}$$

are *sufficient* to determine the values of a, b, c. If the displacements are specified $c = 0$, and we may choose a so that $\psi(z_0) = 0$. Then, by Eq. (5B-5.7), b is defined.

Form of Functions $\psi(z)$ ***and*** $\phi(z)$. By the theory of analytic functions, we know that in a simply connected region D, the analytic functions $\psi(z)$, $\phi(z)$ are single valued and may be represented in the power series (Krrok et al., 1983; Churchill et al., 1989) over R:

$$\begin{aligned} \psi(z) &= \sum_{n=0}^{\infty} a_n z^n \\ \phi(z) &= \sum_{n=0}^{\infty} b_n z^n \end{aligned} \tag{5B-5.9}$$

If the region D is multiply connected, the functions $\psi(z)$, $\phi(z)$ may be multivalued; that is, they may undergo finite incremental changes in traversing a closed contour defining the interior of D (Churchill et al., 1989). Consider for simplicity the doubly connected region R (Fig. 5B-5.1). In circumscribing the boundary C_1, let the functions $\psi(z)$ and $\phi(z)$ receive the increments

$$\begin{aligned} \Delta\psi(z) &= i2\pi\alpha \\ \Delta\phi(z) &= i2\pi\beta \end{aligned} \tag{5B-5.10}$$

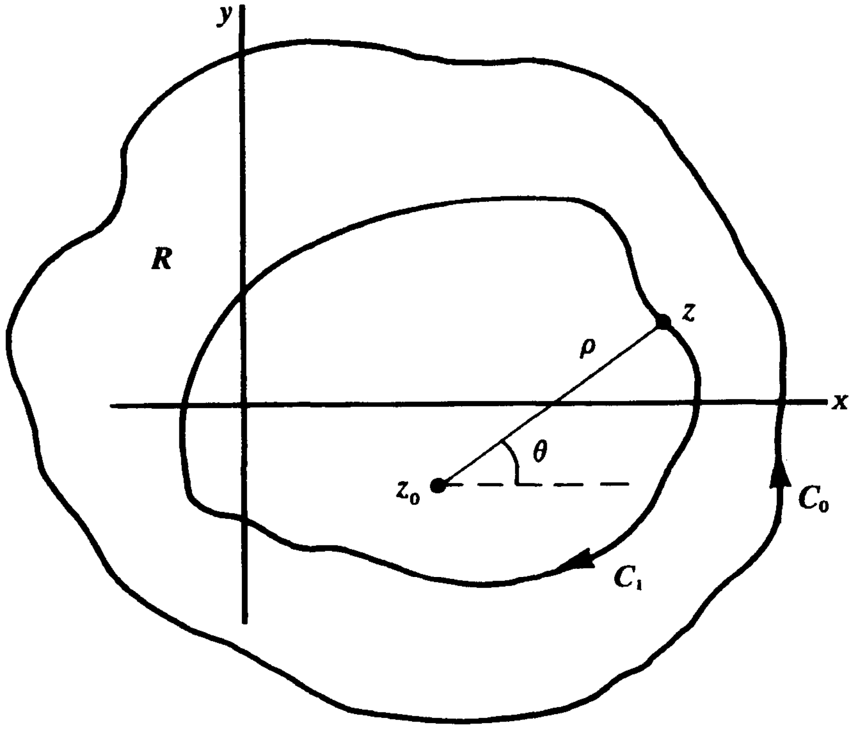


Figure 5B-5.1

where in general (α, β) are complex constants. This type of behavior is exhibited by the function $\log(z - z_0)$, where z_0 is a point inside the contour C_1. Because $z - z_0 = \rho e^{i\theta}$, we have, upon circumscribing C_1,

$$\begin{aligned}\Delta[c \log(z - z_0)] &= [c \log(\rho e^{i\theta})]|_{\rho,0}^{\rho,2\pi} \\ &= c[\log \rho + \log e^{i\theta}]|_{\rho,0}^{\rho,2\pi} \\ &= c[\log \rho + i\theta]|_{\rho,0}^{\rho,2\pi} = i2\pi c \qquad \text{(5B-5.11)}\end{aligned}$$

Hence, we may write

$$\begin{aligned}\psi_0(z) &= \psi(z) - \alpha \log(z - z_0) \\ \phi_0(z) &= \phi(z) - \beta \log(z - z_0)\end{aligned} \qquad \text{(5B-5.12)}$$

where $\psi_0(z)$, $\phi_0(z)$ are analytic within the doubly connected region R, as within R, $\psi_0(z)$ and $\phi_0(z)$ are finite, differentiable, and single-valued. Consequently, we may represent $\psi(z)$ and $\phi(z)$ in the form

$$\begin{aligned}\psi(z) &= \psi_0(z) + \alpha \log(z - z_0) \\ \phi(z) &= \phi_0(z) + \beta \log(z - z_0)\end{aligned} \qquad \text{(5B-5.13)}$$

The requirement that the displacement $u + iv$ be single valued demands that a relation between the constants α and β exist. Thus, substitution of Eqs. (5B-5.13) into the first of Eqs. (5B-5.4) has the result, with the requirement of single-valued displacements, that the term $\kappa\alpha \log(z - z_0) - \overline{\beta \log(z - z_0)}$ vanishes in circumscribing C_1. Hence, noting that $\Delta[\alpha \log(z - z_0)] = 2\pi i\alpha$ and $\Delta[\overline{\beta \log(z - z_0)}] = -2\pi i\bar{\beta}$, we obtain the relation

$$\kappa\alpha + \bar{\beta} = 0 \tag{5B-5.14}$$

Hence, Eqs. (5B-5.13) become

$$\begin{aligned} \psi(z) &= \psi_0(z) + \alpha \log(z - z_0) \\ \phi(z) &= \phi_0(z) - \kappa\bar{\alpha} \log(z - z_0) \end{aligned} \tag{5B-5.15}$$

Finally, we note that $\psi_0(x)$, $\phi_0(z)$ may be represented by the Laurent series (Churchill et al., 1989)

$$\begin{aligned} \psi_0(z) &= \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \\ \phi_0(z) &= \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n \end{aligned} \tag{5B-5.16}$$

as they are analytic in R. Generalization of these results for n-connected regions is given by Muskhelishvili (1975, Chapter 5).

Transformation under Translation and Rotation of Rectilinear Coordinate Axes. For a given state of stress in a plane region, translation of the origin of rectilinear coordinate axes requires that $\psi(z)$ remain invariant, whereas $\chi(z)$ must be modified to maintain the stress state. For a rotation of axes (x, y) into axes (x_1, y_1) through angle α, the functions (ψ, χ) are given by

$$\psi = \psi_1(\zeta)e^{i\alpha}, \qquad \chi = \chi_1(\zeta) \tag{5B-5.17}$$

where $\zeta = ze^{-i\alpha}$ and (ψ_1, χ_1) are functions relative to axes (x_1, y_1), which play the same role as (ψ, χ) relative to axes (x, y) (Muskhelishvili, 1975, p. 137).

5B-6 Plane Elasticity Boundary-Value Problems in Complex Form

As with the three-dimensional theory, we may state the following plane boundary-value problems of elasticity (in the absence of body forces):

1. Determine the states of stress and displacement in region R for given stresses applied to the boundary B of region R.
2. Determine the states of stress and displacement in region R for given displacement of the boundary B of region R.

The uniqueness of solutions of the above problems may be shown (Section 4-16 in Chapter 4) for bounded displacement field and for stress fields that vanish at infinity (Muskhelishvili, 1975).

For the first problem, the plane theory of elasticity is characterized by the equation (in absence of body forces) (see Sections 5-1 and 5-2):

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0\end{aligned}\right\}\quad \text{over } R \tag{5B-6.1}$$

$$\nabla^2(\sigma_x+\sigma_y)=0\}\quad \text{over } R \tag{5B-6.2}$$

$$\left.\begin{aligned}\sigma_{nx}=\sigma_x l+\tau_{xy}m\\ \sigma_{ny}=\tau_{xy}l+\sigma_y m\end{aligned}\right\}\quad \text{on } B \tag{5B-6.3}$$

In terms of the Airy stress function F, we may write these equations as follows (Section 5-4):

$$\sigma_x=\frac{\partial^2 F}{\partial y^2},\qquad \sigma_y=\frac{\partial^2 F}{\partial x^2},\qquad \tau_{xy}=\frac{\partial^2 F}{\partial x\,\partial y} \tag{5B-6.1$'$}$$

$$\nabla^2\nabla^2 F=0 \tag{5B-6.2$'$}$$

$$\begin{aligned}\sigma_{nx}&=\frac{\partial^2 F}{\partial y^2}\ell-\frac{\partial^2 F}{\partial x\,\partial y}m\\ \sigma_{ny}&=-\frac{\partial^2 F}{\partial x\,\partial y}l+\frac{\partial^2 F}{\partial x^2}m\end{aligned} \tag{5B-6.3$'$}$$

Noting the relations (Fig. 5B-6.1)

$$l=\cos\theta=\frac{dy}{ds},\qquad m=\sin\theta=-\frac{dx}{ds} \tag{5B-6.4}$$

we obtain by Eqs. (5B-6.3′) and (5B-6.4) and the chain rule of differentiation [see Eqs. (5B-3.1)]

$$\sigma_{nx}=\frac{d}{ds}\left(\frac{\partial F}{\partial y}\right),\qquad \sigma_{ny}=-\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) \tag{5B-6.5}$$

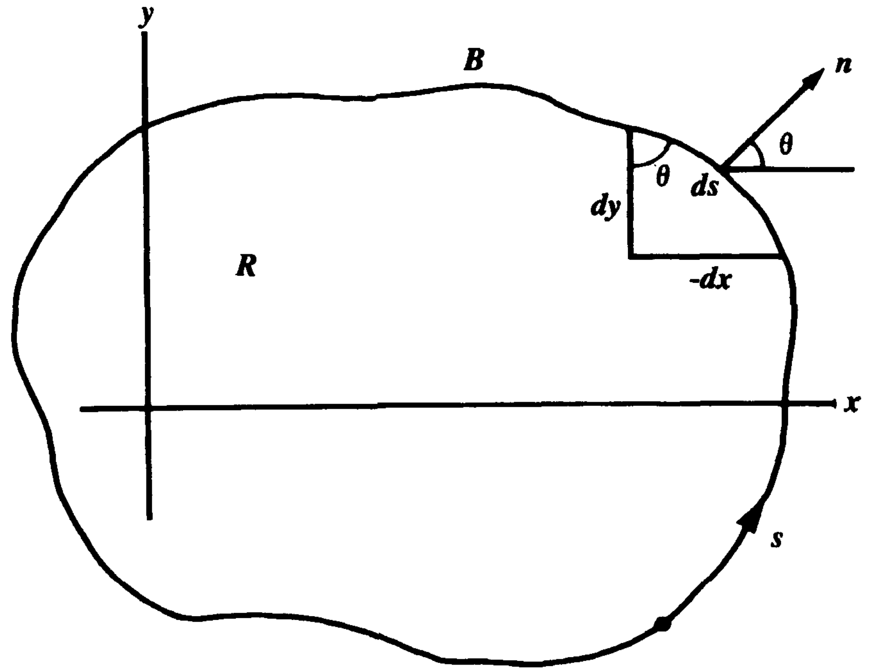


Figure 5B-6.1

Integration of Eqs. (5B-6.5) yields

$$\frac{\partial F}{\partial x} = -\int_B \sigma_{ny}\, ds = f_1(s) + C_1$$
$$\frac{\partial F}{\partial y} = \int_B \sigma_{nx}\, ds = f_2(s) + C_2 \qquad (5B\text{-}6.6)$$

where $f_1(s)$, $f_2(s)$ are functions of s on boundary B and (C_1, C_2) are arbitrary constants. Thus, Eqs. (5B-6.6) define the derivatives of F to within arbitrary constants.

Because Eqs. (5B-6.6) are equivalent to Eqs. (5B-6.3), the first fundamental problem of the plane theory of elasticity may be written in the form

$$\nabla^2\nabla^2 F = \nabla^4 F = 0 \qquad \text{over } R$$
$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= f_1(s) + C_1 \\ \frac{\partial F}{\partial y} &= f_2(s) + C_2 \end{aligned}\right\} \quad \text{on } C \qquad (5B\text{-}6.7)$$

where f_1, f_2 are prescribed functions of s. By Eqs. (5B-2.2), we may write the last two of Eqs. (5B-6.7) in terms of ψ, χ. Thus,

$$\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)} = f_1(s) + if_2(s) + \text{constant} \qquad \text{on } B \tag{5B-6.8}$$

We recall that the first of Eqs. (5B-6.7) is satisfied identically by Eq. (5B-1.3) [or Eq. (5B-1.4)].

For the second fundamental problem, we require that $u = g_1(s)$, $v = g_2(s)$ on B, where (g_1, g_2) are prescribed functions. Hence, for this problem we replace Eq. (5B-6.8) by the boundary condition [see Eq. (5B-2.3)]

$$\kappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)} = 2G(g_1 + ig_2) \qquad \text{on } B \tag{5B-6.9}$$

We have noted the nature of the arbitrariness of functions ψ, χ in Section 5B-5. To within this degree of arbitrariness for the simply connected region, the functions ψ and χ are determined completely by Eqs. (5B-6.8) and (5B-6.9) for the first and second fundamental problems. For details of the mixed fundamental problem (Section 4-15 in Chapter 4), refer to the literature (Muskhelishvili, 1975). For the simply connected region, Eqs. (5B-6.8) [or Eqs. (5B-6.9)] in conjunction with Eqs. (5B-5.9) [or Eqs. (5B-5.13) and (5B-5.16) for the doubly connected region] serve to define $\psi(z)$ and $\phi(z)$, that is, to define the coefficients a_n, b_n [recall $\phi = \psi'(z)$, Eq. (5B-5.1)].

5B-7 Note on Conformal Transformation

Let z and ζ be two complex variables related by the equation

$$z = w(\zeta) \tag{5B-7.1}$$

where $w(\zeta)$ is an analytic function in some domain D in the w plane. Hence, Eq. (5B-7.1) relates every point ζ in the w plane to some definite point in the z plane; that is, Eq. (5B-7.1) defines a one-to-one correspondence between the points in the w plane and the points in the z plane. Also, Eq. (5B-7.1) may be inverted to yield

$$\zeta = f(z) \tag{5B-7.2}$$

Because the points in the z plane cover some region R in the z plane (Fig. 5B-7.1), we say that Eq. (5B-7.1) represents an invertible single-valued "conformal mapping" of region R into the region D (or conversely). The mapping is called conformal because of the following property, which relations of the type of Eq. (5B-7.1) possess where $w(\zeta)$ is analytic: *If in D two line elements emanate from some point ζ and subtend angle θ, then the corresponding elements in R form the same angle, with the sense of θ maintained.* The following discussion depends heavily on topics treated in Churchill et al. (1989).

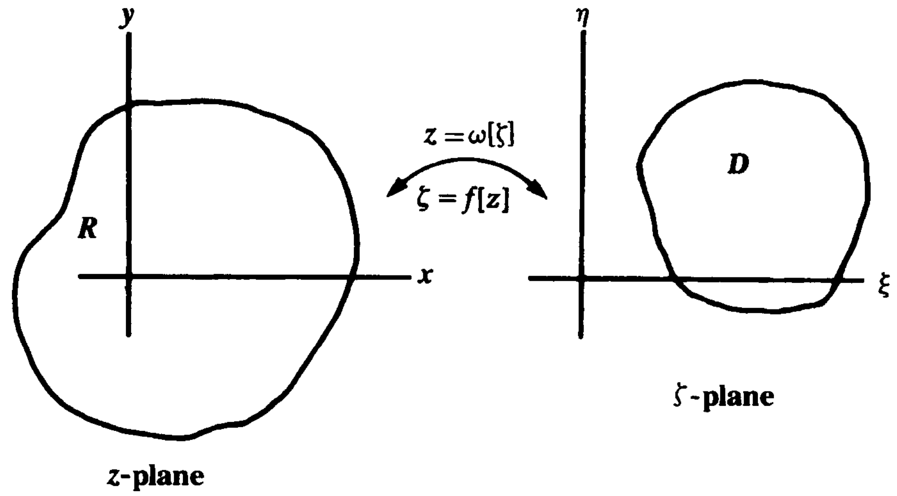


Figure 5B-7.1

Many of the solutions of plane problems of elasticity by the method of complex variables rely heavily on the theorems relative to the unit circle. Thus, fundamental to these solutions is the conformal mapping of a region R in the z plane into a unit circle in the w plane. In particular, two cases are distinguished: (1) the transformation of a simply connected region R interior to a contour C, and (2) the transformation of the region R^* exterior to a contour C (Fig. 5B-7.2).

By the theory of conformal mapping, the transformation (mapping)

$$z = \sum_{k=0}^{\infty} c_k \zeta^k = w(\zeta) \tag{5B-7.3}$$

transforms the interior region R (Fig. 5B-7.2) bounded by the simple contour C (that is, a contour that consists of one closed curve that does not intersect itself) *into* the unit circle. The arbitrary point z_0 can be transformed into an arbitrarily chosen point in the unit circle (say, $\xi = \eta = 0$).

For the region R^* outside contour C, the mapping

$$\begin{aligned} z = w(\zeta) &= \frac{C_{-1}}{\zeta} + \text{an analytic function} \\ &= \frac{C_{-1}}{\zeta} + \sum_{k=0}^{\infty} c_k \zeta^k \end{aligned} \tag{5B-7.4}$$

transforms region R^* (Fig. 5B-7.2) exterior to C *into* the unit circle.

In Eq. (5B-7.3), $w'(\zeta)$, where prime denotes derivative with respect to ζ, has no zero within the unit circle (as it is conformal); hence, $\overline{w'(\zeta)}$ has no zero in D.

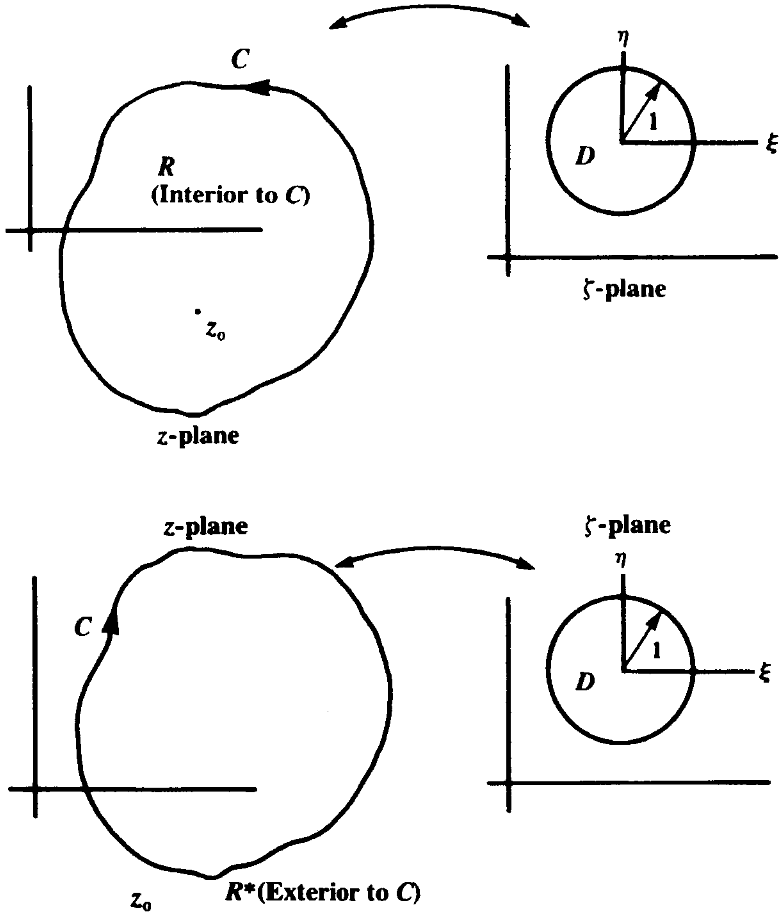


Figure 5B-7.2

Equations (5B-7.3) and (5B-7.4) contain an infinite number of terms in general. However, in practice often only a finite number of terms are used. Hence, instead of transforming the actual region R (or R^*) into the unit circle, an approximation R_a of R is employed. If an exact transformation $w(\zeta)$ is unknown, the coefficients c_n are sometimes determined by methods of the approximate theory of conformal transformations.

Example 5B-7.1. The mapping

$$z = w(\zeta) = -a \int_1^{\zeta} (1 - p^3)^{2/3} \frac{dp}{p^2} + \text{constant} \tag{a}$$

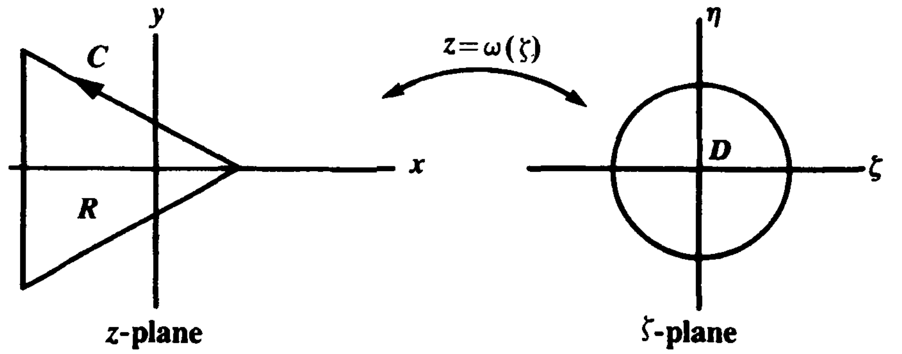


Figure E5B-7.1

where a is a real constant, transforms an equilateral triangle [Fig. (E5B-7.1)] in the z plane into a unit circle in the ζ plane. Noting by the binomial expansion that $(1-p^3)^{2/3} = 1 - \frac{2}{3}p^3 + \frac{1}{9}p^6 - \frac{4}{27}p^9 + \cdots$, and choosing the constant in Eq. (a) properly, we find

$$z = w(\zeta) = -a\left(\frac{1}{\zeta} + \frac{1}{3}\zeta^2 + \frac{1}{45}\zeta^5 + \cdots\right) \tag{b}$$

For the boundary of the unit circle, $\zeta = 1e^i$. Thus, for the contour of region R, Eq. (b) yields

$$z = -a(e^{-i\theta} + \tfrac{1}{3}e^{2i\theta} + \tfrac{1}{45}e^{5i\theta} + \cdots) \tag{c}$$

Approximations R_a to the equilateral triangle (region R) may be obtained by taking $2, 3, 4, \ldots$ terms in Eq. (c). With three terms, a fairly good approximation to the equilateral triangle is obtained.

Curvilinear Coordinates in the Plane. Because much of the complex variable method relates to the conformal mapping of a given region R in the z plane into a region D (unit circle) in the ζ plane, it is natural to introduce polar coordinates (r, θ) in the ζ plane (see Chapter 6). Then, $\zeta = \xi + i\eta$, where $\xi = r\cos\theta$, $\eta = r\sin\theta$ may be written as $\zeta = re^{i\theta}$. Hence,

$$z = x + iy = w(\zeta) = w(re^{i\theta}) \tag{5B-7.5}$$

Accordingly, the circles $r =$ constant and the radii $\theta =$ constant in the ζ plane are transformed into orthogonal curvilinear coordinate lines (a, b) in the z plane by Eq. (5B-7.5) (see Section 1-20, Chapter 1), as $z = w(\zeta)$ is a conformal transformation (Fig. 5B-7.3). The tangents to the coordinate lines are denoted by the symbols A, B and form a base for the axes of the curvilinear coordinate system at point z_0. Because the transformation is conformal, the axes (A, B) are right-handed (conform to axes

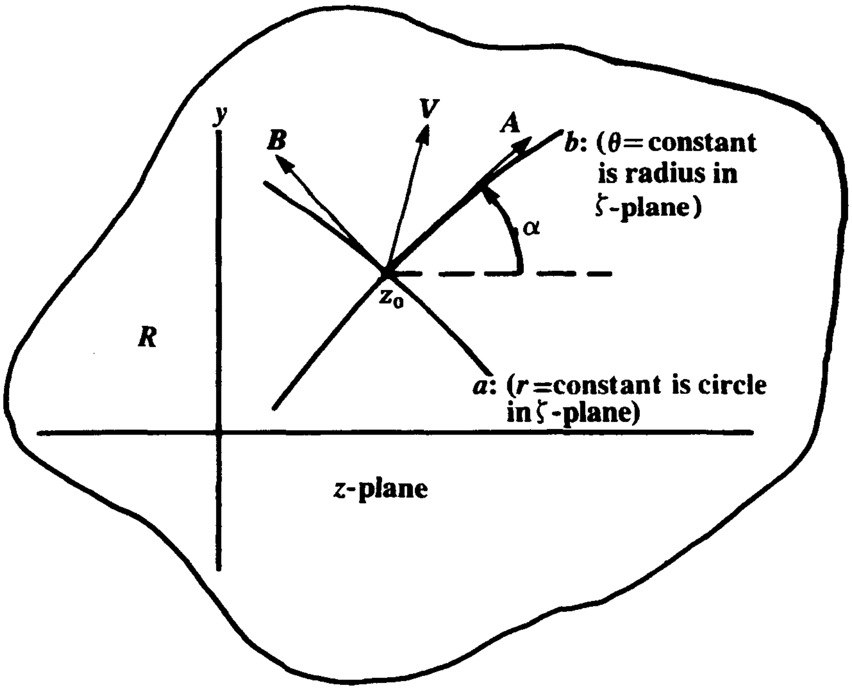


Figure 5B-7.3

x, y) as a conformal transformation preserves the orientation of directions. The axis A forms the angle α with respect to the x direction.

In the sequel we require expressions for the transformations of displacement components (u, v), which are vectors. Accordingly, consider a vector **V** in the z plane at the point $z = w(re^{i\theta})$. By Fig. 5B-7.3 we find

$$V_x + iV_y = (V_A + iV_B)e^{i\alpha} \tag{5B-7.6}$$

Equation (5B-7.6) relates the components (V_x, V_y) relative to the (x, y) axes to the components (V_A, V_B) relative to curvilinear coordinates (a, b). To express $e^{i\alpha}$ in terms of the transformation $z = w(\zeta)$, we note that if we consider a displacement dz of the point z in the direction of the tangent A, the corresponding point ζ (in D) will undergo displacement $d\zeta$ in the radial direction (θ = constant). Thus,

$$dz = e^{i\alpha}|dz|, \qquad d\zeta = e^{i\theta}|d\zeta|$$

and, with Eq. (5B-7.5),

$$\begin{aligned} e^{i\alpha} &= \frac{dz}{|dz|} = \frac{w'(\zeta)\, d\zeta}{|w'(\zeta)|\, |d\zeta|} = e^{i\theta}\, \frac{w'(\zeta)}{|w'(\zeta)|} \\ &= \frac{\zeta}{r}\, \frac{w'(\zeta)}{|w'(\zeta)|} \end{aligned} \tag{5B-7.7}$$

Equations (5B-7.6) and (5B-7.7) yield

$$V_x + iV_y = (V_A + iV_B)\frac{\zeta}{r}\frac{w'(\zeta)}{|w'(\zeta)|}$$

or

$$V_A + iV_B = e^{-i\alpha}(V_x + iV_y) = \frac{\bar{\xi}}{r}\frac{\overline{w'(\zeta)}}{|w'(\zeta)|} \qquad \text{(5B-7.8)}$$

where $\bar{\xi} = re^{-i\theta}$ and $\overline{w'(\zeta)}$ are complex conjugates of ζ and $w'(\zeta)$.

Problems

1. Let $z = c\cosh\zeta$, where $z = x + iy$, $\zeta = \xi + i\eta$. Derive the equations that define the coordinate lines in the z plane that correspond to the coordinate lines $\xi = \xi_0 =$ constant, $\eta = \eta_0 =$ constant in the ζ plane. Show that the coordinate lines form an orthogonal system.
2. Let $z = ia\coth(\zeta/2)$, where $z = x + iy$, $\zeta = \xi + i\eta$. Repeat Problem 1.

5B-8 Plane Elasticity Formulas in Terms of Curvilinear Coordinates

To transform the stress components and the displacement components to curvilinear coordinates (a, b), we must transform $\psi(z)$, $\chi(z)$ into functions of ζ, that is, into functions $\psi(\zeta)$, $\chi(\zeta)$, where $z = w(\zeta)$.

Stress Components. Let σ_a, σ_b, τ_{ab} be defined as follows (Fig. 5B-8.1):

$\sigma_a =$ normal stress component on curve $a =$ constant

$\sigma_b =$ normal stress component on curve $b =$ constant

$\tau_{ab} = \tau_{ba} =$ shear component on both curves

By plane transformation laws of stress (Section 3-7 in Chapter 3), we obtain

$$\begin{aligned}\sigma_a &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \tau_{ab} &= -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha + \tau_{xy}\cos 2\alpha\end{aligned} \qquad \text{(5B-8.1)}$$

Letting $\alpha \to \alpha + \pi/2$, we obtain from the first of Eq. (5B-8.1)

$$\sigma_b = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha - \tau_{xy}\sin 2\alpha \qquad \text{(5B-8.2)}$$

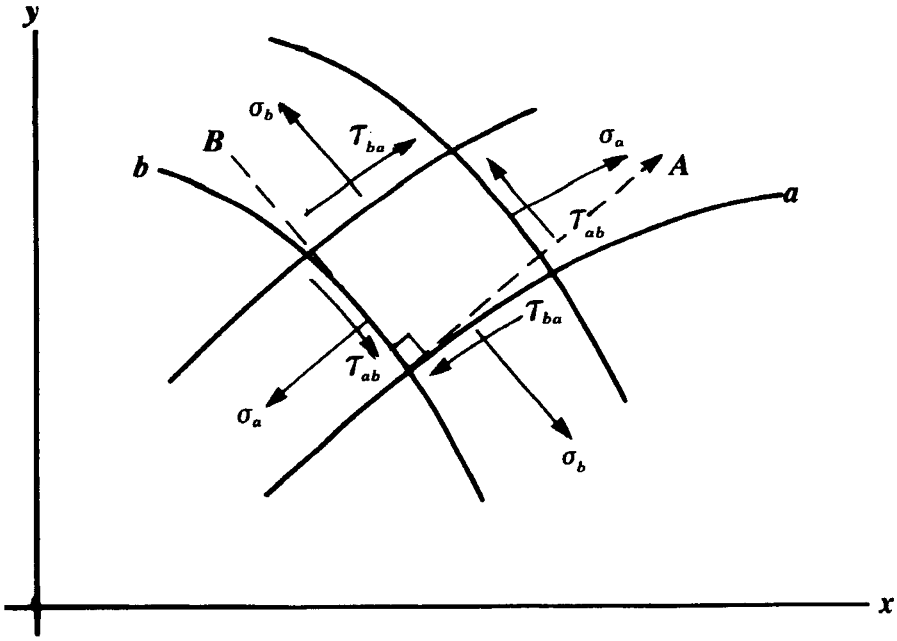


Figure 5B-8.1

Hence, by Eqs. (5B-8.1) and (5B-8.2), we find

$$\sigma_a + \sigma_b = \sigma_x + \sigma_y \tag{5B-8.3}$$

$$\sigma_b - \sigma_a + 2i\tau_{ab} = e^{2i\alpha}(\sigma_y - \sigma_x + 2i\tau_{xy}) \tag{5B-8.4}$$

To obtain an expression for the term $e^{2i\alpha}$, we note by Eq. (5B-7.7) that

$$\begin{aligned} e^{2i\alpha} &= \frac{\zeta^2}{r^2}\frac{[w'(\zeta)]^2}{|w'(\zeta)|^2} = \frac{\zeta^2}{r^2}\frac{[w'(\zeta)]^2}{w'(\zeta)\overline{w'(\zeta)}} \\ &= \frac{\zeta^2}{r^2}\frac{w'(\zeta)}{\overline{w'(\zeta)}} \end{aligned} \tag{5B-8.5}$$

Thus, Eqs. (5B-8.4) and (5B-8.5) yield

$$\begin{aligned} \sigma_b - \sigma_a + 2i\tau_{ab} &= \frac{\zeta^2}{r^2}\frac{w'(\zeta)}{\overline{w'(\zeta)}}(\sigma_y - \sigma_x + 2i\tau_{xy}) \\ \sigma_a + \sigma_b &= \sigma_x + \sigma_y \end{aligned} \tag{5B-8.6}$$

To express σ_a, σ_b in terms of ζ, we note that by Eqs. (5B-3.6) and (5B-7.5)

$$\sigma_x + \sigma_y = 2[\psi'(z) + \overline{\psi'(z)}] = 2\left[\frac{\psi_1'(\zeta)}{w'(\zeta)} + \frac{\overline{\psi_1'(\zeta)}}{\overline{w'(\zeta)}}\right] \tag{5B-8.7}$$

where $\psi_1(\zeta) = \psi(z)$. In a similar manner, we may express $\sigma_b - \sigma_a + 2i\tau_{ab}$ in terms of ζ.

Displacement Components. Let (u, v) denote the (x, y) components of displacement (Fig. 5B-8.2). Let (u_a, u_b) denote the (a, b) components of displacement. Then, by vector projections, we find

$$(u_a + u_b) = e^{-i\alpha}(u + iv) \tag{5B-8.8}$$

where $(u + iv)$ is expressed in terms of ψ and χ by Eq. (5B-2.3), which in turn may be expressed in terms of ζ.

Equations (5B-8.6) and (5B-8.8) express the stress components and the displacement components of plane elasticity in curvilinear plane coordinates in the z plane (polar coordinates r, θ in the ζ plane).

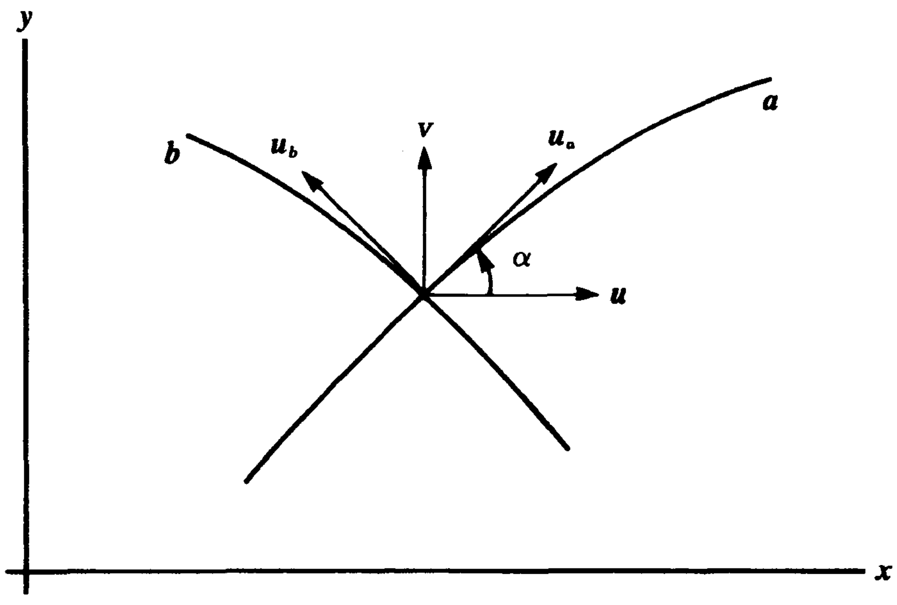


Figure 5B-8.2

5B-9 Complex Variable Solution for Plane Region Bounded by Circle in the z Plane

In this section we demonstrate the complex variable method for the case of a simply connected circular region R in the z plane, with prescribed boundary stresses on the circle C (Fig. 5B-9.1). The case of prescribed displacement on C may be treated in an analogous manner. Although the example is elementary, the essential features of the complex variable method are illustrated. (The more complicated problem of the plane region with circular hole is treated in Section 6-10 of Chapter 6.)

Solution Relative to z. We take axes (x, y) with origin at the center of the circle C. We consider the components of the boundary stress $(\sigma_{nr}, \sigma_{n\alpha})$ on C to be known, continuous, and single-valued functions of α on C. Accordingly, by Eqs. (5B-3.3) and (5B-6.8), we have (with constant = 0)

$$f_1(s) + if_2(s) = i\int_0^s (\sigma_{nx} + i\sigma_{ny})\, ds = ia\int_0^\alpha (\sigma_{nx} + i\sigma_{ny})\, d\alpha \tag{5B-9.1}$$

Overall equilibrium of region R requires

$$\sum F_x = a\int_0^{2\pi} \sigma_{nx}\, d\alpha = a\int_0^{2\pi} (\sigma_{nr}\cos\alpha - \sigma_{n\alpha}\sin\alpha)\, d\alpha = 0 \tag{5B-9.2}$$

$$\sum F_y = a\int_0^{2\pi} \sigma_{ny}\, d\alpha = a\int_0^{2\pi} (\sigma_{nr}\sin\alpha + \sigma_{n\alpha}\cos\alpha)\, d\alpha = 0 \tag{5B-9.3}$$

$$\sum M_0 = a\int_0^{2\pi} \sigma_{n\alpha}\, d\alpha = 0 \tag{5B-9.4}$$

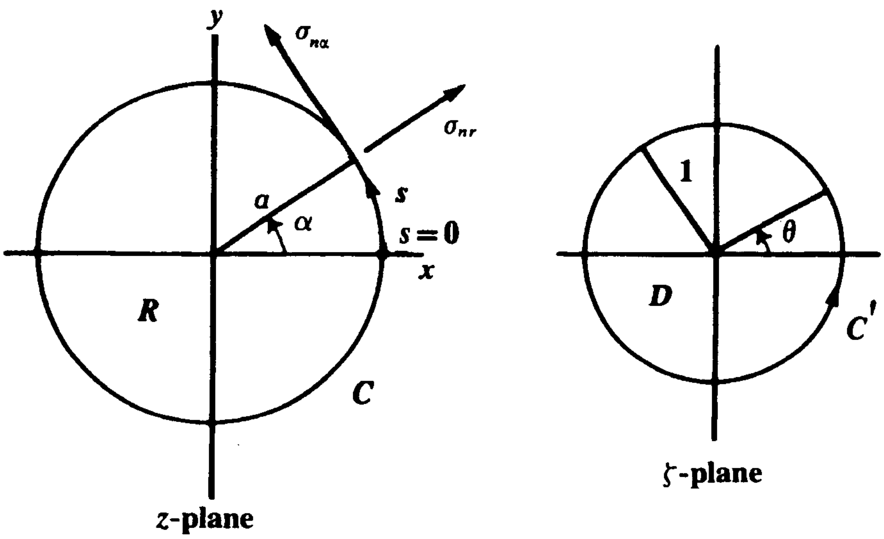


Figure 5B-9.1

Hence, $(\sigma_{nr}, \sigma_{n\alpha})$ are periodic in α (with period 2π). Assuming $(\sigma_{nr}, \sigma_{n\alpha})$ are continuous, single-valued functions of α (Dirichlet conditions; Krrok et al., 1983), we may represent them (and hence σ_{nx}, σ_{ny}) in the form of convergent Fourier series. Thus, we may express $f_1 + if_2$ in the known form

$$f_1 + if_2 = \sum_{m=-\infty}^{m=\infty} A_m e^{im\alpha} \tag{5B-9.5}$$

where by Fourier series theory

$$A_m = \frac{1}{2\pi}\int_0^{2\pi} (f_1 + if_2)e^{-im\alpha}\, d\alpha \tag{5B-9.6}$$

By Section 5B-5 [Eq. (5B-5.9)], we have in R, $|z| < a$ for analytic functions ψ, χ'

$$\begin{aligned} \psi(z) &= \sum_{n=1}^{\infty} a_n z^n \\ \chi'(z) = \phi(z) &= \sum_{n=0}^{\infty} b_n z^n \end{aligned} \tag{5B-9.7}$$

where we have taken $\psi(0) = 0$. Assuming that the series of Eqs. (5B-9.7) converge in R and on C, we have by Eq. (5B-6.8) and (5B-9.7)

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n a^n e^{in\alpha} + \bar{a}_1 a e^{i\alpha} \\ &\quad + \sum_{n=0}^{\infty}(n+2)\bar{a}_{n+2}a^{n+2}e^{-in\alpha} + \sum_{n=0}^{\infty} \bar{b}_n a^n e^{-in\alpha} \\ &\quad = \sum_{m=-\infty}^{\infty} A_m e^{im\alpha} \end{aligned} \tag{5B-9.8}$$

where we note the formulas

$$z = ae^{i\alpha}, \qquad \bar{z} = ae^{-i\alpha}$$

$$\overline{\psi'(z)} = \sum_{n=1}^{\infty} n\bar{a}_n \bar{z}^{n-1}, \qquad \overline{\phi(z)} = \overline{\chi'(z)} = \sum_{n=0}^{\infty} \bar{b}_n z^{-n}$$

$$\begin{aligned} z\sum_{n=1}^{\infty} n\bar{a}_n \bar{z}^{n-1} &= \sum_{n=1}^{\infty} n\bar{a}_n a^n e^{-(n-2)i\alpha} \\ &= \bar{a}_1 a e^{i\alpha} + \sum_{n=0}^{\infty}(n+2)\bar{a}_{n+2}a^{n+2}e^{-in\alpha} \end{aligned} \tag{5B-9.9}$$

Comparing like powers of e in Eq. (5B-9.8), we obtain

$$
\begin{aligned}
&e^{i\theta}: \quad a(a_1 + \bar{a}_1) = A_1 = \text{real number} \qquad (n = 1) \\
&e^{in\theta}: \quad a_n = A_n \qquad (n > 1) \\
&e^{-in\theta}: \quad (n+2)\bar{a}_{n+2}a^{n+2} + a^n\bar{b}_n = -A_{-n} \qquad (n > 0)
\end{aligned}
\tag{5B-9.10}
$$

Equations (5B-9.10) define all the coefficients a_n, b_n except a_1. Only the real value of a_1 is defined by the first of Eqs. (5B-9.10), as $a_1 + \bar{a}_1 = \text{Re}(a_1)$. However, this condition is sufficient, because the imaginary part of $\psi'(z)$ may be chosen arbitrarily for $z = 0$. For example, we may take $\text{Im}\ \psi'(0) = 0$ [Eq. (5B-5.8)]. Furthermore, the constant A_1 has the physical significance that it is the average (mean) value of radial load acting on the boundary C or R. This result follows from Eqs. (5B-9.6) and (5B-9.1). Thus, by Eq. (5B-9.6),

$$
\begin{aligned}
2\pi A_1 &= \int_0^{2\pi} (f_1 + if_2)e^{-i\alpha}\, d\alpha \\
&= \int_0^{2\pi} (f_1 \cos\alpha + f_2 \sin\alpha)\, d\alpha + i\int_0^{2\pi} (f_2 \cos\alpha - f_1 \sin\alpha)\, d\alpha
\end{aligned}
$$

However, we note that by Eqs. (5B-9.2)–(5B-9.4),

$$
\begin{aligned}
\sum M_0 &= a\int_0^{2\pi} \sigma_{n\alpha}\, d\alpha = a\int_0^{2\pi} (\sigma_{ny}\cos\alpha - \sigma_{nx}\sin\alpha)\, d\alpha \\
&= -\int_0^{2\pi} (\cos\alpha\, df_1 + \sin\alpha\, df_2) \\
&= -[f_1 \cos\alpha + f_2 \sin\alpha]_0^{2\pi} + \int_0^{2\pi} (-f_1 \sin\alpha + f_2 \cos\alpha)\, d\alpha \\
&= \int_0^{2\pi} (-f_1 \sin\alpha + f_2 \cos\alpha)\, d\alpha = 0
\end{aligned}
$$

and in a similar manner

$$
\begin{aligned}
\sum F_r &= a\int_0^{2\pi} \sigma_{nr}\, d\alpha = a\int_0^{2\pi} (\sigma_{nx}\cos\alpha + \sigma_{ny}\sin\alpha)\, d\alpha \\
&= \int_0^{2\pi} (f_1 \cos\alpha + f_2 \sin\alpha)\, d\alpha
\end{aligned}
$$

Hence,

$$A_1 = \frac{a}{2\pi}\int_0^{2\pi} \sigma_{nr}\, d\alpha \tag{5B-9.11}$$

and A_1 equals the mean value of the radial load.

Solution Relative to ζ. Alternatively, the solution may be derived in the ζ plane (Fig. 5B-9.1). For example, the region R in the z plane may be transformed into the unit circle D in the ζ plane by the mapping

$$z = w(\zeta) = a\zeta \tag{5B-9.12}$$

Hence, on the boundary C',

$$\begin{gathered} \zeta = e^{i\theta} = \gamma \\ \frac{w(\zeta)}{\overline{\overline{w'(\zeta)}}} = \frac{a\zeta}{a} = \zeta(= \gamma \text{ on } C') \end{gathered} \tag{5B-9.13}$$

To write boundary conditions on C', we require that $\psi(z)$ and $\chi'(z)$ be transformed into functions of ζ. For this purpose, we remark that with the notation

$$\psi_1(\zeta) = \psi(z) = \psi[w(\zeta)]$$
$$\phi_1(\zeta) = \phi(\zeta) = \phi[w(\zeta)]$$

we have

$$\psi'(z) = \frac{d\psi}{dz} = \frac{d\psi_1(\zeta)}{d\zeta}\frac{d\zeta}{dz} = \psi'_1/w'(\zeta)$$

Consequently, the boundary conditions [Eq. (5B-6.8)] in terms of $\zeta(= \gamma \text{ on } C')$ become, with $\phi(z) = \chi'(z)$,

$$\psi_1(\gamma) + \frac{w(\gamma)}{\overline{w'(\gamma)}}\,\overline{\psi'_1(\gamma)} + \overline{\phi_1(\gamma)} = f_1 + if_2 \tag{5B-9.14}$$

or with Eqs. (5B-9.13)

$$\psi_1(\gamma) + \gamma\overline{\psi'_1(\gamma)} + \overline{\phi_1(\gamma)} = f_1 + if_2 \tag{5B-9.15}$$

With

$$\psi_1(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n, \qquad \phi_1(\zeta) = \sum b_n \zeta^n \tag{5B-9.16}$$

the analysis proceeds as in the z plane [following Eqs. (5B-9.7)]. Then substitution of $\psi(z)$, $\phi(z)$ [or $\psi_1(\zeta)$, $\phi_1(\zeta)$] into expressions for $\sigma_x + \sigma_y$, $\sigma_y - \sigma_x + 2i\tau_{xy}$, $2G(u + iv)$ yields (x, y) components in the z plane (in the ζ plane), provided ψ, ϕ are absolutely and uniformly convergent on the boundary circle $|z| = a$. If first derivatives of σ_{nr}, $\sigma_{n\theta}$ (or σ_{nx}, σ_{ny}) satisfy Dirichlet conditions (Churchill et al., 1989), this requirement is satisfied.

Problem Set 5B

1. Consider the problem of small deflections, plane thermoelasticity for which

$$\epsilon_x = \gamma_{xz} = \gamma_{yz} = 0$$

(a) Derive an expression for σ_z in terms of stress components σ_x and σ_y, material properties k (thermal coefficients of linear expansion) and E (modulus of elasticity), and temperature change T measured from an arbitrary zero.

(b) Assume the additional conditions that stress components $\sigma_x = \sigma_y = \tau_{xy} = 0$. Hence, derive expressions for the strain components ϵ_x, ϵ_y and γ_{xy}.

(c) Show that under the combined conditions of parts (a) and (b), the compatibility conditions reduce to $\nabla^2 T = 0$ for constant E and k.

(d) Using the results of part (b), show that the rotation of a volume element in the xy plane is

$$\omega_z = \partial v/\partial x = -\partial u/\partial y$$

Hence, show that

$$\partial\epsilon'/\partial x = \partial\omega_x/\partial y, \qquad \partial\epsilon'/\partial y = -\partial\omega_x/\partial x$$

where $\epsilon' = (1 + \nu)kT$. That is, show that ϵ' and ω_z satisfy the Cauchy–Riemann equations. (Consequently, the theory associated with the Cauchy–Riemann equations may be applied to ϵ' and ω_z.)

2. Let z denote the complex variable $z = x + iy$, where (x, y) denote plane rectangular Cartesian coordinates. Let $\bar{z} = x - iy$ denote the complex conjugate of z.

(a) Show that the equilibrium equations of plane elasticity in the absence of body forces may be transformed into the result ($i^2 = -1$)

$$\frac{\partial}{\partial z}(\sigma_x - \sigma_y + 2i\tau_{xy}) + \frac{\partial}{\partial \bar{z}}(\sigma_x + \sigma_y) = 0$$

(b) Let the displacement s be given by $s = u + iv$ where (u, v) denotes (x, y) displacement components. Express $\partial s/\partial \bar{z}$ in terms i and derivatives of u, v relative to (x, y).

(c) Noting that the equilibrium equation of part (a) is a necessary and sufficient condition that there exists a function $F(z, \bar{z})$ such that

$$\frac{\partial F}{\partial z} = \sigma_x + \sigma_y, \qquad \frac{\partial F}{\partial \bar{z}} = \sigma_y - \sigma_x - 2i\tau_{xy}$$

and expressing $(\sigma_x, \sigma_y, \tau_{xy})$ in terms of (u, v), show for plane strain, employing the results of part (b), that

$$4Gs = -F(z, \bar{z}) + f(z)$$

where $2G(1 + \nu) = E$. E = Young's modulus, ν = Poisson's ratio.

(d) Compute the derivative $\partial s/\partial z$ in terms of i and derivatives of (u, v) with respect to (x, y). Hence, show that

$$\sigma_x + \sigma_y - f'(z) = -4G\,\frac{\partial s}{\partial z} = \frac{\partial F}{\partial z} - f'(z)$$

$$4(\lambda + G)\,\frac{\partial s}{\partial z} = \sigma_x + \sigma_y + 2i(\lambda + G)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

where

$$\lambda = \nu E/[(1 + \nu)(1 - 2\nu)].$$

3. Show that the equation $\sigma_x + \sigma_y = 4\mathrm{Re}[\psi'(z)]$ may be written in the form

$$\sigma_x + \sigma_y + 4iE\omega/[(1 + \nu)(1 + \kappa)] = 4\psi'(z)$$

where ω is the volumetric rotation.

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