CHAPTER 6

PLANE ELASTICITY IN POLAR COORDINATES

The use of polar coordinates is advantageous in problems involving boundaries formed by circular arcs or radially straight lines. Furthermore, certain problems of symmetry lend themselves well to polar coordinates. Accordingly, in this chapter we express the basic plane-elasticity equations in polar coordinates.

6-1 Equilibrium Equations in Polar Coordinates

Consider an element of volume bounded by the polar coordinate lines (r, θ) and $(r+dr,\ \theta+d\theta)$ (Fig. 6-1.1). Let the thickness h of the element [dimension perpendicular to the (x, y) plane] be a function of (r, θ). Let the element be subjected to stress as shown (R and Θ denote body forces per unit volume in the radial and tangential directions, respectively). Because $d\theta$ is an infinitesimal angle, summations of forces in the radial and tangential directions yield for equilibrium, assuming that the thickness is sufficiently small compared to the in-plane dimensions so that variations of radial and tangential stresses over the thickness can be neglected,

$$\begin{aligned}
&\frac{\partial(h\sigma_r)}{\partial r}+\frac{1}{r}\frac{\partial(h\tau_{r\theta})}{\partial\theta}+\frac{h(\sigma_r-\sigma_\theta)}{r}+hR=0\\
&\frac{\partial(h\tau_{r\theta})}{\partial r}+\frac{1}{r}\frac{\partial(h\sigma_\theta)}{\partial\theta}+\frac{2(h\tau_{r\theta})}{r}+h\Theta=0
\end{aligned}\tag{6-1.1}$$

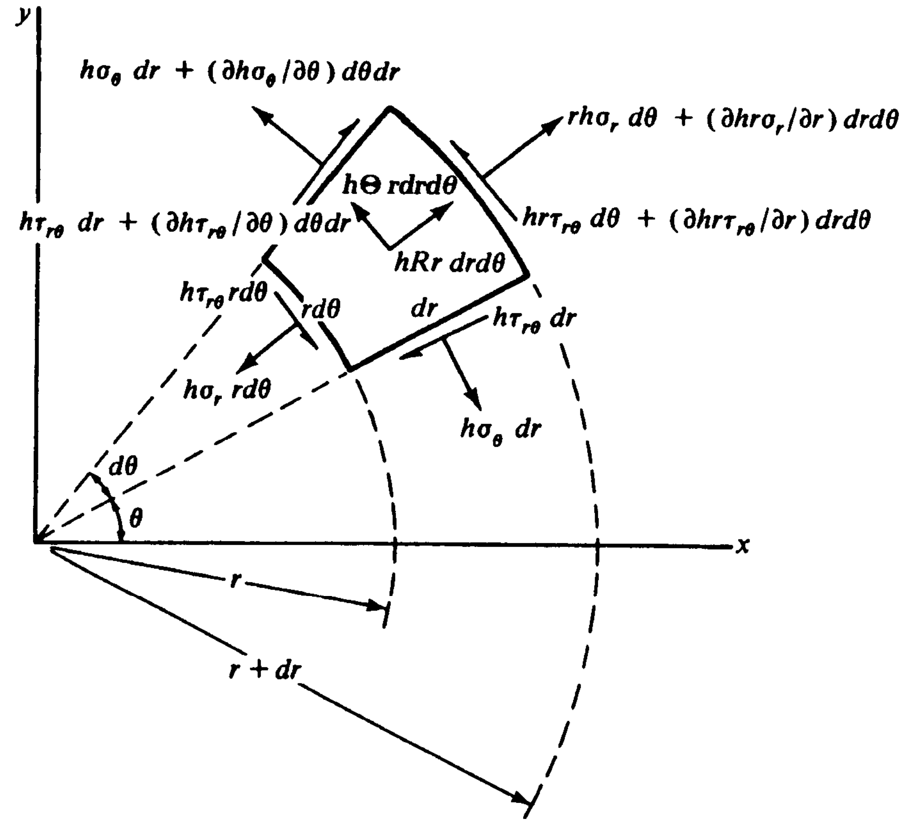


Figure 6-1.1

Equations (6-1.1) are the equilibrium equations for plane elasticity in polar coordinates. They are equivalent to Eqs. (5-2.11) in Chapter 5. Alternatively, Eqs. (6-1.1) may be derived by mathematically transforming Eqs. (5-2.11) from (x, y) coordinates to (r, θ) coordinates by tensor theory (see also Appendix 3A in Chapter 3). For h = constant h may be canceled from Eqs. (6-1.1).

6-2 Stress Components in Terms of Airy Stress Function $F = F(r, \theta)$

To derive stress components in terms of the Airy stress function F, where F is considered to be a function of polar coordinates (r, θ), we may transform Eqs. (5-4.3) (for constant thickness and in the absence of body forces) to polar coordinates as follows. By Fig. (6-1.1), we obtain the following relations between (x, y) and (r, θ):

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x &= r\cos\theta, \qquad y = r\sin\theta \\ \tan\theta &= \frac{y}{x} \end{aligned} \tag{6-2.1}$$

Consider first the transformation of σ_x. By Eq. (5-4.9), we note that we require $\partial^2 F/\partial y^2$ in terms of (r, θ). By the chain rule of partial differentiation and Eq. (6-2.1), we have

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial F}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial F}{\partial \theta}\cos\theta$$

Similarly,

$$\begin{aligned}\frac{\partial^2 F}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial F}{\partial y}\\ &= \frac{\partial^2 F}{\partial r^2}\sin^2\theta + \frac{2}{r}\frac{\partial^2 F}{\partial r\,\partial\theta}\sin\theta\cos\theta - \frac{2}{r^2}\frac{\partial F}{\partial \theta}\sin\theta\cos\theta\\ &\quad + \frac{1}{r}\frac{\partial F}{\partial r}\cos^2\theta + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\cos^2\theta\end{aligned}$$

Now, noting that as $\theta \to 0$, $\sigma_x \to \sigma_r$, $\cos\theta \to 1$, $\sin\theta \to 0$, we obtain

$$\sigma_r = \left.\frac{\partial^2 F}{\partial y^2}\right|_{\theta\to 0} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}$$

Also, noting that as $\theta \to \pi/2$, $\sigma_y \to \sigma_\theta$, $\cos\theta \to 0$, $\sin \to 1$, we find

$$\sigma_\theta = \left.\frac{\partial^2 F}{\partial y^2}\right|_{\theta\to \pi/2} = \frac{\partial^2 F}{\partial r^2}$$

In a similar manner, we may evaluate $\partial^2 F/\partial x\,\partial y$. Then, noting that as $\theta \to 0, \tau_{xy} \to \tau_{r\theta}$, we find

$$\tau_{r\theta} = -\left.\frac{\partial^2 F}{\partial x\,\partial y}\right|_{\theta\to 0} = -\frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial F}{\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)$$

Accordingly, the stress components are given in terms of the Airy stress function $F(r, \theta)$ by the relations

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\\ \sigma_\theta &= \frac{\partial^2 F}{\partial r^2}\\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)\end{aligned} \qquad (6\text{-}2.2)$$

More generally, the preceding transformations may be carried out with respect to orthogonal curvilinear coordinates (Section 1-22). For variable thickness $h = h(r, \theta)$, we replace σ_r, σ_θ, $\tau_{r\theta}$ in Eq. (6-2.2) by $h\sigma_r$, $h\sigma_\theta$, $h\tau_{r\theta}$ [see Eqs. (6-1.1)]. Also, for certain cases, body forces may be introduced simply (see Section 6-6).

6-3 Strain–Displacement Relations in Polar Coordinates

Consider a point P in a medium that undergoes a deformation (Fig. 6-3.1). Under the deformation, the point P moves to P^*. With respect to rectangular Cartesian coordinates (x, y), the displacement components of point P are (u, v); with respect to polar coordinates, the displacement components are (U, V). Accordingly, by Fig. 6-3.1,

$$
\begin{aligned}
u &= U\cos\theta - V\sin\theta \\
v &= U\sin\theta + V\cos\theta
\end{aligned}
\tag{6-3.1}
$$

Substitution of Eqs. (6-3.1) into Eq. (2-15.14) yields ϵ_x, ϵ_y, γ_{xy} in terms of U, V, and θ. For example, consider ϵ_x. By Eqs. (2-15.14) and the chain rule for partial differentiation, we obtain

$$
\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}
\tag{6-3.2}
$$

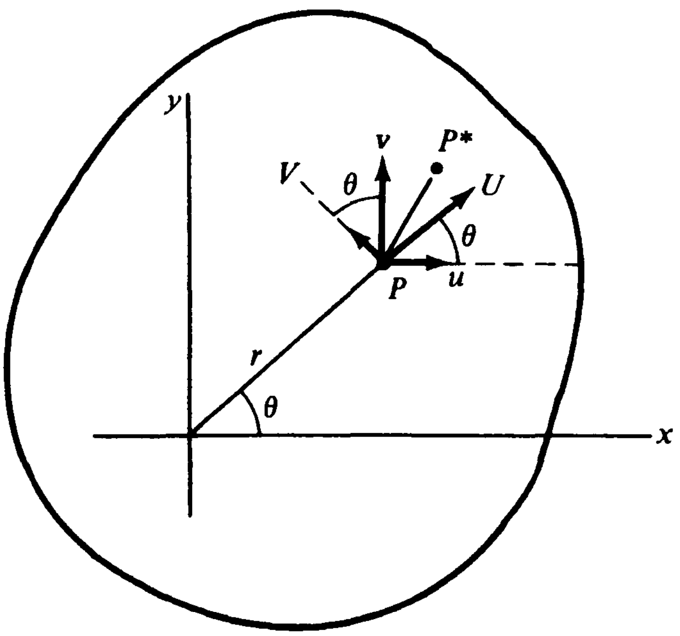


Figure 6-3.1

where, by Eqs. (6-3.1) and (6-2.1),

$$
\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial U}{\partial \theta}\cos\theta - U\sin\theta - \frac{\partial V}{\partial \theta}\sin\theta - V\cos\theta \\
\frac{\partial u}{\partial r} &= \frac{\partial U}{\partial r}\cos\theta - \frac{\partial V}{\partial r}\sin\theta \\
\frac{\partial \theta}{\partial x} &= -\frac{\sin\theta}{r} \\
\frac{\partial r}{\partial x} &= \cos\theta
\end{aligned}
\tag{6-3.3}
$$

Accordingly, by Eqs. (6-3.2) and (6-3.3), we obtain

$$
\begin{aligned}
\epsilon_x =& \left(-\frac{\partial U}{\partial \theta}\cos\theta + U\sin\theta + \frac{\partial V}{\partial \theta}\sin\theta + V\cos\theta\right)\frac{\sin\theta}{r} \\
&+ \left(\frac{\partial U}{\partial r}\cos\theta - \frac{\partial V}{\partial r}\sin\theta\right)\cos\theta
\end{aligned}
$$

Noting that $\epsilon_x \to \epsilon_r$, $\sin\theta \to 0$, and $\cos\theta \to 1$ as $\theta \to 0$, we obtain

$$
\epsilon_r = \epsilon_x|_{\theta\to 0} = \frac{\partial U}{\partial r}
$$

Analogously, $\epsilon_x \to \epsilon_\theta$, $\sin\theta \to 1$, and $\cos\theta \to 0$ as $\theta \to \pi/2$. Hence,

$$
\epsilon_\theta = \epsilon_x|_{\theta\to\pi/2} = \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r}
$$

Finally, in a similar manner, we may express γ_{xy} as a function of U, V, and θ, and noting that $\gamma_{xy} \to \gamma_{r\theta}$ as $\theta \to 0$, we obtain

$$
\gamma_{r\theta} = \gamma_{xy}|_{\theta\to 0} = \frac{\partial V}{\partial r} - \frac{V}{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}
$$

Accordingly, the strain components ϵ_r, ϵ_θ, $\gamma_{r\theta}$ with respect to polar coordinates (r, θ) are

$$
\begin{aligned}
\epsilon_r &= \frac{\partial U}{\partial r} \\
\epsilon_\theta &= \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \theta} \\
\gamma_{r\theta} &= \frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r}
\end{aligned}
\tag{6-3.4}
$$

where $U = U(r, \theta)$, $V = V(r, \theta)$ are the radial and tangential displacement components (Fig. 6-3.1).

Alternatively, Eq. (6-3.4) may be derived by the method of Section 2-6 in Chapter 2 (see also Appendix 2B).

Problem. Derive the last of Eqs. (6-3.4).

With the understanding that (u, v) denote radial and tangential components of displacement relative to (r, θ) coordinates, we may write

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r}, \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \end{aligned} \tag{6-3.5}$$

The strain–compatibility relations in polar coordinates may be obtained either by elimination of (u, v) from Eqs. (6-3.5) or by transformation of Eqs. (5-3.1) in Chapter 5 into polar coordinates. Thus, for plane deformations we obtain the compatibility relation

$$\frac{\partial}{\partial r}\left(r\frac{\partial \gamma_{r\theta}}{\partial \theta} - r^2\frac{\partial \epsilon_\theta}{\partial r}\right) + r\frac{\partial \epsilon_r}{\partial r} - \frac{\partial^2 \epsilon_r}{\partial \theta^2} = 0 \tag{6-3.6}$$

For the special case of rotationally symmetric problems where all quantities are functions of radial coordinate r only, by Eqs. (6-3.5), we obtain the compatibility relations

$$\begin{aligned} \epsilon_r &= \frac{d}{dr}(r\epsilon_\theta) \\ \gamma_{r\theta} &= r\frac{d}{dr}\left(\frac{v}{r}\right) \end{aligned} \tag{6-3.7}$$

Problem Set 6-3

1. Consider two orthogonal line elements, ds_1 and ds_2, one radial and one tangential in a plane R (Fig. P6-3.1). Consider the following separate deformations: (a) all points in the body (region) undergo a radial displacement; $U = U_1(r, \theta)$, $V = V_1 = 0$, where (U, V) denote radial and tangential components of displacement; (b) all points undergo a displacement such that $U = U_2 = 0$, $V = V_2(r, \theta)$. Derive expressions for the strain components ϵ_r, ϵ_θ, $\gamma_{r\theta}$ corresponding to the deformations (a) and (b). Superimpose the results of deformations (a) and (b) to arrive at Eqs. (6-3.4).

2. The line t is tangent to the centerline of a circular arc ring AB at point P (see Fig. P6-3.2). When the ring is loaded, point P undergoes radial and tangential displacement components (w, u). Derive an expression for $\tan(\phi^* - \phi)$, the tangent of the angle through which line t rotates. Linearize this formula for small rotations, that is, for $\tan(\phi^* - \phi) \approx \phi^* - \phi$.

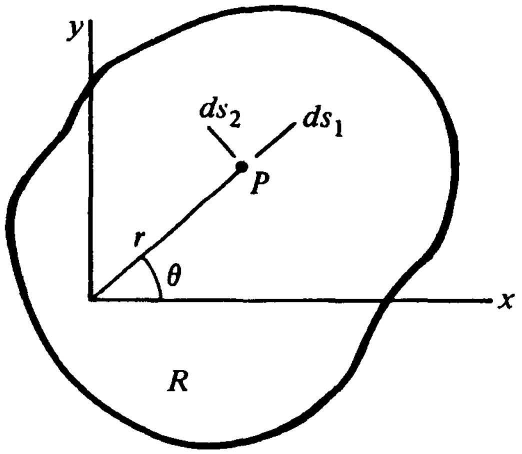


Figure P6-3.1

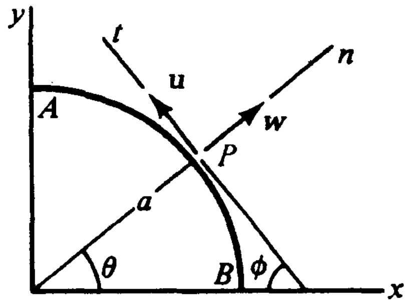


Figure P6-3.2

Recall that $\tan(\phi^* - \phi) = (\tan\phi^* - \tan\phi)/(1 + \tan\phi^* \tan\phi)$. Note that $u = u(\theta)$, $w = w(\theta)$. *Express the results in terms of a, w, u and derivatives of w and u.*

6-4 Stress–Strain–Temperature Relations

Equations (5-2.12) and (5-2.13) in Chapter 5 remain valid for any orthogonal plane coordinates, except that the derivatives $\partial/\partial x$, $\partial/\partial y$ must be transformed appropriately. Accordingly, relative to polar coordinates (r, θ), we have the stress–strain relations

$$\begin{aligned} \sigma_r &= \lambda e + 2G\epsilon_r \\ \sigma_\theta &= \lambda e + 2G\epsilon_\theta \\ \tau_{r\theta} &= G\gamma_{r\theta} \\ e &= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \end{aligned} \tag{6-4.1}$$

where (u, v) are displacement components relative to polar coordinates (r, θ); see Fig. 6-3.1 (where U, V are used).

Accordingly, for plane strain we have the stress–strain–temperature relations [Eqs. (5-3.8)]

$$\begin{aligned}
\sigma_r &= \frac{E}{(1+v)(1-2v)}[(1-v)\epsilon_r + v\epsilon_\theta - (1+v)kT] \\
\sigma_\theta &= \frac{E}{(1+v)(1-2v)}[v\epsilon_r + (1-v)\epsilon_\theta - (1+v)kT] \\
\tau_{r\theta} &= \frac{E}{2(1+v)}\gamma_{r\theta} \\
\sigma_z &= \frac{E}{(1+v)(1-2v)}[v(\epsilon_r + \epsilon_\theta) - (1+v)kT] \\
e &= \epsilon_r + \epsilon_\theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\epsilon_z &= \gamma_{rz} = \gamma_{\theta z} = \tau_{rz} = \tau_{\theta z} = 0
\end{aligned} \tag{6-4.2}$$

and for the compatibility relations in terms of stress components [Eq. (5-3.9)]

$$\nabla^2(\sigma_r + \sigma_\theta) + \frac{E}{1-v}\nabla^2(kT) + \frac{1}{1-v}\left(\frac{\partial B_r}{\partial r} + \frac{1}{r}B_r + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right) = 0 \tag{6-4.3}$$

where (B_r, B_θ) denote body forces relative to (r, θ) coordinates, T denotes temperature, and k is the coefficient of linear thermal expansion. For plane stress, we have the stress–strain–temperature relations [Eq. (5-3.10)]

$$\begin{aligned}
\sigma_r &= \frac{E}{1-v^2}[\epsilon_r + v\epsilon_\theta - (1+v)kT] \\
\sigma_\theta &= \frac{E}{1-v^2}[v\epsilon_r + \epsilon_\theta - (1+v)kT] \\
\tau_{r\theta} &= \frac{E}{2(1+v)}\gamma_{r\theta} \\
\epsilon_z &= -\frac{1}{1-v}[v(\epsilon_r + \epsilon_\theta) - (1+v)kT] \\
e &= \epsilon_r + \epsilon_\theta + \epsilon_z = \frac{1}{1-v}[(1-2v)(\epsilon_r + \epsilon_\theta) + (1+v)kT] \\
\sigma_z &= \tau_{rz} = \tau_{\theta z} = \gamma_{rz} = \gamma_{\theta z} = 0
\end{aligned} \tag{6-4.4}$$

and the compatibility relations [Eq. (5-3.11)]

$$\nabla^2(\sigma_r + \sigma_\theta) + E\nabla^2(kT) + (1+v)\left(\frac{\partial B_r}{\partial r} + \frac{1}{r}B_r + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right) = 0 \tag{6-4.5}$$

Problem Set 6-4

1. **(a)** For the case of plane stress relative to the (x, y) plane, write the integral V of the strain energy density U in terms of rectangular Cartesian coordinates (x, y). Neglect temperature effects.
   (b) Express the integral V in terms of polar coordinates (r, θ).
   (c) Derive expressions for the stress components relative to polar coordinates (Section 4-3 in Chapter 4).
2. A circular ring, with rectangular cross section, has a unit thickness perpendicular to its plane. Its inner boundary $(r = a)$ is fixed. Its outer boundary $(r = b)$ is subjected to a uniform shearing stress S directed in the counterclockwise sense. (a) In terms of polar coordinates (r, θ), with origin at the center of the ring, and polar coordinate stress components, write the integral V for the strain energy density U of the ring (plane stress case).
   It may be shown that the stress solution for this problem is given by $\sigma_r = \sigma_\theta = 0$, $\tau_{r\theta} = Sb^2/r^2$. (b) Evaluate the integral V of the strain energy density U. (c) By equating V to the work done during loading (the shear stress at $r = b$ is increased from zero to S), compute the rotation of the ring at $r = b$.
3. In Problem 2, determine the tangential (θ) displacement v as a function of r, where $v = 0$ at $r = a$, and $\tau_{r\theta} = S$ at $r = b$.
4. In Problem 2, assume $\sigma_r = \sigma_\theta = u = 0$, where u is the radial displacement. Show that $\tau_{r\theta} = Sb^2/r^2$. (Assume $v = 0$ at $r = a$ and $\tau_{r\theta} = S$ at $r = b$.)

6-5 Compatibility Equation for Plane Elasticity in Terms of Polar Coordinates

Expressing the second derivative of F with respect to x in terms of polar coordinates and adding it to the second derivative of F with respect to y derived in Section 6-2, we obtain

$$\sigma_x + \sigma_y = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \tag{6-5.1}$$

Also, by Eqs. (6-2.2) we note that

$$\sigma_r + \sigma_\theta = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \tag{6-5.2}$$

Accordingly, by Eqs. (6-5.1), (6-5.2), and (5-7.1), we obtain the compatibility relation (for constant body forces, or body forces derivable from a potential function) in terms of polar coordinates (r, θ):

$$\nabla^2\nabla^2 F = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \tag{6-5.3}$$

Accordingly, in polar coordinates [see Sectin 1-22 and Eq. (1-22.13) in Chapter 1]

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{6-5.4}$$

A solution of the compatibility equation $\nabla^2\nabla^2 F = 0$ in polar coordinates was derived by J. H. Michell (1899) for a certain class of plane problems. A modified form of the solution given by Michell[1] is

$$\begin{aligned} F &= A_0 \log r + B_0 r^2 + C_0 r^2 \log r + D_0 r^2 \theta + A_0' \theta \\ &\quad + \frac{A_1}{2} r\theta \sin\theta + (B_1 r^3 + A_1' r^{-1} + B_1' r \log r)\cos\theta \\ &\quad - \frac{C_1}{2} r\theta \cos\theta + (D_1 r^3 + C_1' r^{-1} + D_1' r \log r)\sin\theta \\ &\quad + \sum_{n=2}^{\infty}(A_n r^n + B_n r^{n+2} + A_n' r^{-n} + B_n' r^{-n+2})\cos n\theta \\ &\quad + \sum_{n=2}^{\infty}(C_n r^n + D_n r^{n+2} + C_n' r^{-n} + D_n' r^{-n+2})\sin n\theta \end{aligned} \tag{6-5.5}$$

Problem Set 6-5

1. Consider a ring loaded as shown in Fig. P6-5.1. Show that the function

$$\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\cos 2\theta + Fr^2 + H \log r$$

satisfies $\nabla^2\nabla^2\phi = 0$. Determine the constants A, B, C, D, F, H to satisfy the stress boundary conditions. Hence, derive formulas for σ_r, σ_θ, $\tau_{r\theta}$.

2. Derive the equation of compatibility for plane problems in polar coordinates in terms of the strain components [see Eq. (6-3.6)].

[1] The term $D_0 r^2 \theta$ was not given by Michell. Also, Michell included the terms $r\cos\theta$, $r\sin\theta$, which are not included here. However, these terms yield zero stress components. See S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed. (New York: McGraw-Hill Book Company, 1970), Chapter 4. See also A. Timpe, *Z. Math. Phys.* 52: 348 (1905) and *Math. Z.*, 17; 189 (1923).

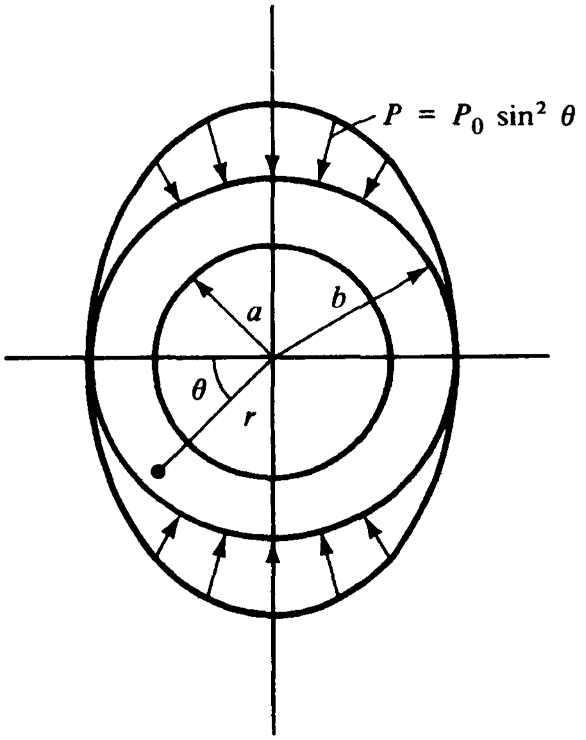


Figure P6-5.1

3. The stress function $F = (E\delta/4\pi)r\ \log\ r \sin\theta$ has been proposed as a possible solution for a circular ring with a radial slit (Fig. P6-5.3), where δ is the radial displacement at the slit.
   (a) Write down a complete set of boundary conditions in terms of stress components and displacement components.
   (b) Outline a procedure to determine a stress function that satisfies *all* boundary conditions.

4. For a problem of plane stress,

$$Eu = (1 - \nu)(\log r)\cos\theta - 2\cos\theta + 2\theta\sin\theta$$
$$Ev = (1 - \nu)(1 - \log r)\sin\theta + 2\theta\cos\theta$$

   where (u, v) are displacement components in polar coordinates (r, θ), E is the modulus of elasticity, and ν is Poisson's ratio. There is no body force.
   (a) Is this a possible displacement vector if the origin is included in the body? Explain.
   (b) Is this a possible displacement vector for a closed ring with center at the origin? Explain.
   (c) Does the corresponding Airy stress function satisfy the compatibility condition $\nabla^2\nabla^2 F = 0$? Explain.
   (d) Show that for this problem the stress components σ_r and σ_θ are equal.

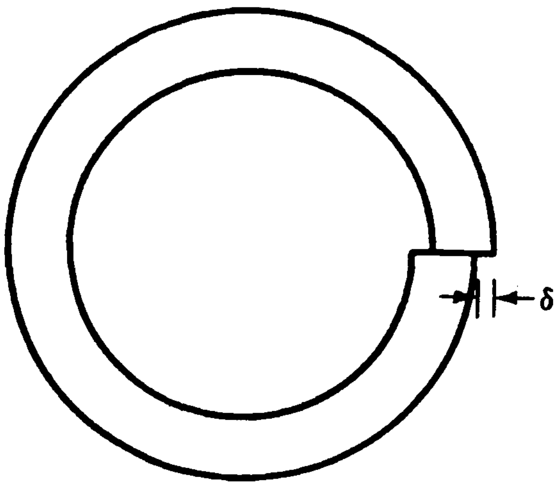


Figure P6-5.3

5. In addition to the terms given in Eq. (6-5.5) (obtained by the method of separation of variables), the terms

$$F_1 = A\theta r^2 \log r, \qquad F_2 = B\theta \log r$$
$$F_3 = C\theta r \cos\theta \log r, \qquad F_4 = D\theta r \sin\theta \log r$$

are also solutions to the biharmonic equation of plane elasticity, in the absence of body forces and thermal effects. Discuss the application of these terms to regions R_1, R_2, R_3, with polar coordinate systems shown in Fig. P6-5.5.

6-6 Axially Symmetric Problems

For axially symmetric problems, $F = F(r)$. Then the equilibrium equations [see Eqs. (6-1.1)] reduce to (for h = constant)

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) + R = 0, \qquad \Theta = 0 \tag{6-6.1}$$

Accordingly, for axially symmetric problems of equilibrium the tangential body force Θ is zero, and the two stress components $(\sigma_r, \sigma_\theta)$ and the radial body force R are functions of r only. Furthermore, the shearing stress $\tau_{r\theta}$ [see Eqs. (6-2.2)] is zero.

The compatibility relation simplifies to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr}\right) = 0 \tag{6-6.2}$$

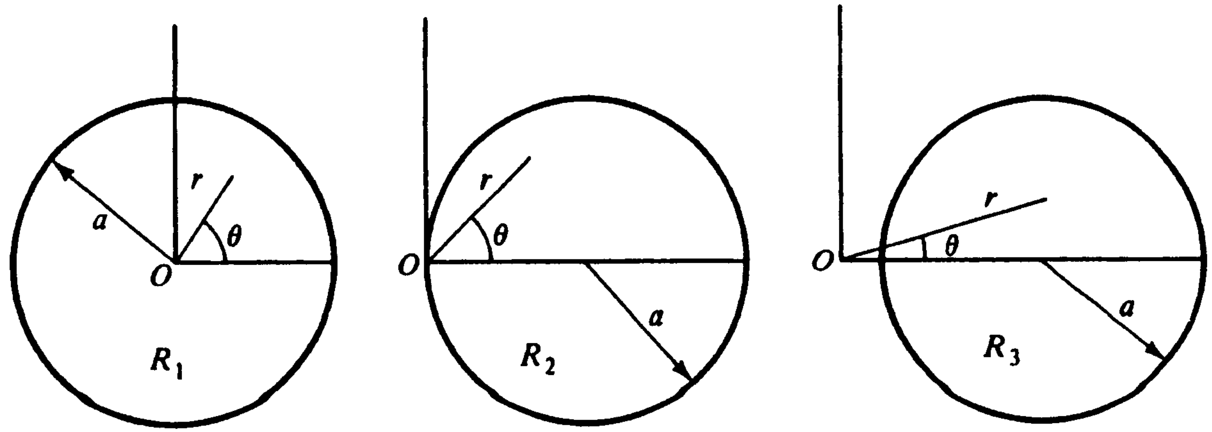


Figure P6-5.5

Equation (6-4.2) may be written in the form

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)\right]\right\} = 0 \tag{6-6.3}$$

In this latter form, the Airy stress function F may be determined by direct integration. Accordingly, for problems of axial symmetry, integration of Eq. (6-6.3) yields the Airy stress function in the form

$$F = A\log r + Br^2\log r + Cr^2 + D \tag{6-6.4}$$

where A, B, and C are arbitrary constants of integration, which are determined by boundary conditions. The constant D does not enter into the formulas for the stress components, as they depend on derivatives of F. Thus, by Eqs. (6-2.2) and Eq. (6-6.4), we obtain

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{dF}{dr} = \frac{A}{r^2} + B(1 + 2\log r) + 2C \\ \sigma_\theta &= \frac{d^2F}{dr^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C\end{aligned} \tag{6-6.5}$$

For a doubly connected region bounded by contours L_1 and L_2 and with the origin of coordinates (r, θ) inside the inner contour (Fig. 6-6.1), the requirement that the displacement be single valued dictates that $B = 0$ (See Example 6-6.2; see also remarks at the end of Section 5-4 in Chapter 5.)

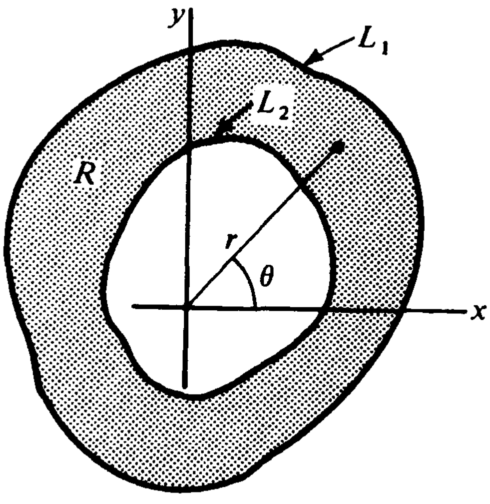


Figure 6-6.1

Inclusion of Body Forces. A direct and elementary treatment of the most generally rotationally symmetric plane state of stress for linear isotropic elastic materials under arbitrary body forces has been given by Stern (1965). The main results follow.

For the most general rotationally symmetric plane problem, relative to polar coordinates (r, θ) we assume that the stress components are independent of θ. Thus, Eqs. (6-1.1), with h = constant, yield

$$\begin{aligned} \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \\ \frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} + \Theta = 0 \end{aligned} \tag{6-6.6}$$

Recallying Eq. (5-3.12) and expressing ∇^2 and $\partial/\partial x$, $\partial/\partial y$ in terms of (r, θ), in the absence of temperature effects, we obtain for the equation of compatibility

$$\frac{d^2}{dr^2}(\sigma_r + \sigma_\theta) + \frac{1}{r}\frac{d}{dr}(\sigma_r + \sigma_\theta) = -K_2\left(\frac{dR}{dr} + \frac{R}{r}\right) \tag{6-6.7}$$

where for plane strain $K_2 = 1/(1 - \nu)$ and for plane stress $K_2 = 1 + \nu$. For purposes of integration, it is convenient to rewrite Eqs. (6-6.6) and (6-6.7) in the forms

$$\frac{1}{r^2}\frac{d}{dr}(r^2\sigma_r) = \frac{1}{r}(\sigma_r + \sigma_\theta) - R \tag{6-6.8}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{r\theta}) = -\Theta \tag{6-6.9}$$

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[\frac{d}{dr}(\sigma_r + \sigma_\theta) + K_2 R\right]\right\} = 0 \tag{6-6.10}$$

where by the assumption of independency of θ, R and Θ must be independent of θ.

Integration of Eq. (6-6.10) yields

$$\sigma_r + \sigma_\theta = A \log r + B - K_2 H(r) \tag{6-6.11}$$

where A and B are constants to be defined by the boundary conditions, and

$$H(r) = \int_{r_0}^{r} R(\xi)\, d\xi \tag{6-6.12}$$

where r_0 is some fixed (arbitrary) value of r. Hence, by Eqs. (6-6.8) and (6-6.11), we find, after integration, that

$$\sigma_r = \frac{C}{r^2} + \frac{A}{4}[2\log(r) - 1] + \frac{B}{2} - \frac{K_2}{2}H(r) + \frac{K_2 - 2}{2}I(r) \tag{6-6.13}$$

where C is a constant of integration and

$$I(r) = \frac{1}{r^2}\int_{r_0}^{r} \xi^2 R(\xi)\, d\xi \tag{6-6.14}$$

By Eqs. (6-6.11) and (6-6.13), we obtain

$$\sigma_\theta = -\frac{C}{r^2} + \frac{A}{4}[2\log(r) + 1] + \frac{B}{2} - \frac{K_2}{2}H(r) + \frac{2 - K_2}{2}I(r) \tag{6-6.15}$$

Finally, integration of Eq. (6-6.9) yields

$$\tau_{r\theta} = \frac{D}{r^2} - J(r) \tag{6-6.16}$$

where D is a constant and

$$J(r) = \frac{1}{r^2}\int_{r_0}^{r} \xi^2 \Theta(\xi)\, d\xi \tag{6-6.17}$$

The displacement components are obtained by integrating the strain–displacement relations. However, the displacement need not be rotationally symmetric. Let u and v denote the radial and transverse components of displacement. Then the strain–displacement relations in conjunction with Hooke's law give for plane stress [see Eq. (5-3.10) with $kT = 0$]

$$\epsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E}\sigma_r - \frac{\nu}{E}\sigma_\theta \tag{6-6.18}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}\sigma_\theta - \frac{\nu}{E}\sigma_r \tag{6-6.19}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = \frac{2(1+\nu)}{E}\tau_{r\theta} \tag{6-6.20}$$

With the aid of Eqs. (6-6.13) and (6-6.15) and integration by parts, Eq. (6-6.18) yields

$$u(r,\theta) = -\frac{1+\nu}{E}\frac{C}{r} + \frac{1-\nu}{2E}Ar\log r - \frac{3-\nu}{4E}Ar + \frac{1-\nu}{2E}Br - \frac{1-\nu^2}{2E}r[H(r)-I(r)] + f(\theta)$$

where $f(\theta)$ is an undetermined function of θ only. Putting this result in Eq. (6-6.20) and noting Eq. (6-6.16), we can write

$$v(r,\theta) = -\frac{1+\nu}{E}\frac{D}{r} - \frac{1+\nu}{E}r[G(r)-J(r)] + \frac{df}{d\theta} + rg(\theta)$$

where $g(\theta)$ is another undetermined function and

$$G(r) = \int_{r_0}^{r} \Theta(\xi)\, d\xi$$

As a consequence of Eq. (6-6.19), however, we conclude that

$$\frac{d^2f}{d\theta^2} + f = 0, \qquad \frac{dg}{d\theta} = \frac{A}{E}$$

so that finally we find

$$u(r,\theta) = -\frac{1+\nu}{E}\frac{C}{r} + \frac{1-\nu}{2E}Ar\log r - \frac{3-\nu}{4E}Ar + \frac{1-\nu}{2E}Br - \frac{1-\nu^2}{2E}r[H(r)-I(r)] + M\cos\theta + N\sin\theta$$

$$v(r,\theta) = -\frac{1+\nu}{E}\frac{D}{r} + \frac{A}{E}r\theta - \frac{1+\nu}{E}r[G(r)-J(r)] - M\sin\theta + N\cos\theta + Lr$$

where the constants M and N represents the Cartesian components of a rigid-body translation and L is a rigid-body rotation angle.

In certain cases, restrictions may be imposed on the constants. For example, we should note that if the origin is contained in the body, then the constants A, C, and D must necessarily vanish. The constant A must also vanish whenever the origin can be encircled by a contour entirely in the body, even though the origin itself is not; this guarantees single-valued displacements. Finally, if any portion of the body extends indefinitely, the constant A must vanish for stresses to remain bounded.

The accelerating disk affords a rather simple application of the preceding results. Consider a circular disk of radius b clamped to a rotating shaft on a concentric circular portion of the disk of radius $a, 0 < a < b$. We suppose that at some particular instant the shaft is rotating with angular velocity ω and angular acceleration α. In a quasi-static analysis the problem may be rephrased as a circular ring clamped along the inner boundary $r = a$ and free of traction on the outer boundary $r = b$, and further subjected to the body-force densities

$$R = \rho r \omega^2, \qquad \Theta = -\rho r \alpha$$

where ρ is the mass density of the disk, assumed uniform throughout. Integrating from the inner boundary ($r_0 = a$), we obtain

$$\begin{aligned}
H(r) &= \frac{1}{2}\rho\omega^2(r^2 - a^2) \\
I(r) &= \frac{\rho\omega^2}{4r^2}(r^4 - a^4) \\
G(r) &= -\frac{1}{2}\rho\alpha(r^2 - a^2) \\
J(r) &= -\frac{\rho\alpha}{4r^2}(r^4 - a^4)
\end{aligned}$$

Because the ring is complete, $A = 0$. Furthermore, on the outer boundary $\sigma_r = \tau_{r\theta} = 0$. Hence,

$$\begin{aligned}
\frac{C}{b^2} + \frac{B}{2} - \frac{\rho\omega^2}{8b^2}(b^2 - a^2)[(3 + \nu)b^2 + (1 - \nu)a^2] = 0 \\
\frac{D}{b^2} + \frac{\rho\alpha}{4b^2}(b^4 - a^4) = 0
\end{aligned}$$

At $r = a$, $u = v = 0$, so that $M = N = 0$ and

$$\begin{aligned}
-\frac{1 + \nu}{E}\frac{C}{a} + \frac{1 - \nu}{E}\frac{aB}{2} = 0 \\
-\frac{1 + \nu}{E}\frac{D}{a} + aL = 0
\end{aligned}$$

Thus, we find

$$\begin{aligned}
\frac{1}{2}B &= \frac{(1 + \nu)\rho\omega^2}{8}K \\
C &= \frac{(1 - \nu)\rho\omega^2 a^2}{8}K \\
D &= -\frac{\rho\alpha}{4}(b^4 - a^4) \\
L &= -\frac{(1 + \nu)\rho\alpha}{4Ea^2}(b^4 - a^4)
\end{aligned}$$

where

$$K = (b^2 - a^2)\frac{(3+\nu)b^2 + (1-\nu)a^2}{(1+\nu)b^2 + (1-\nu)a^2}$$

Then the stresses are given by

$$\sigma_r = \frac{\rho\omega^2}{8}\left\{\left[1 + \frac{a^2}{r^2}\right]K - \left[1 - \frac{a^2}{r^2}\right][(3+\nu)r^2 + (1-\nu)a_2 - \nu K]\right\}$$

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left\{\left[1 + \frac{a^2}{r^2}\right]\nu K - \left[1 - \frac{a^2}{r^2}\right][(1+3\nu)r^2 - (1-\nu)a^2 - K]\right\}$$

$$\tau_{r\theta} = -\frac{q\alpha}{4r^2}(b^4 - r^4)$$

while the displacement components are simply

$$u = \frac{\rho\omega^2(1-\nu^2)}{8E}r\left[1 - \frac{a^2}{r^2}\right][K - (r^2 - a^2)]$$

$$v = -\frac{\rho\alpha(1+\nu)}{4E}r\left[1 - \frac{a^2}{r^2}\right]\left[\frac{b^4}{a^2} - r^2\right]$$

Example 6-6.1. Let $A = B = 0$ in Eq. (6-6.5). Then Eq. (6-6.5) yields

$$\sigma_r = \sigma_\theta = 2C \tag{a}$$

Equation (a) represents the case of constant stress throughout the plane [see Fig. E6-6.1].

Example 6-6.2. Let $B = 0$ in Eqs. (6-6.5). Then

$$\sigma_r = \frac{A}{r^2} + 2C, \qquad \sigma_\theta = -\frac{A}{r^2} + 2C \tag{b}$$

Equation (b) may be used to represent the stress in a thick-walled cylinder with inner radius a and outer radius b and with internal pressure p_i and external pressure p_0 (Fig. E6-6.2.1.) Then the boundary conditions are

$$\begin{aligned} \sigma_r &= -p_0 \quad \text{for} \quad r = b \\ \sigma_r &= -p_i \quad \text{for} \quad r = a \end{aligned} \tag{c}$$

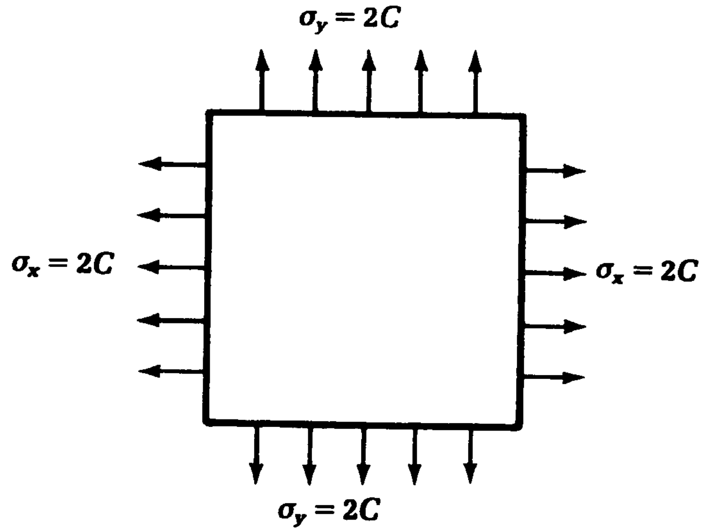


Figure E6-6.1

Substitution of Eqs. (c) into Eqs. (b) yields

$$A = \frac{a^2 b^2 (p_0 - p_i)}{b^2 - a^2}$$
$$2C = \frac{p_i a^2 - p_0 b^2}{b^2 - a^2} \tag{d}$$

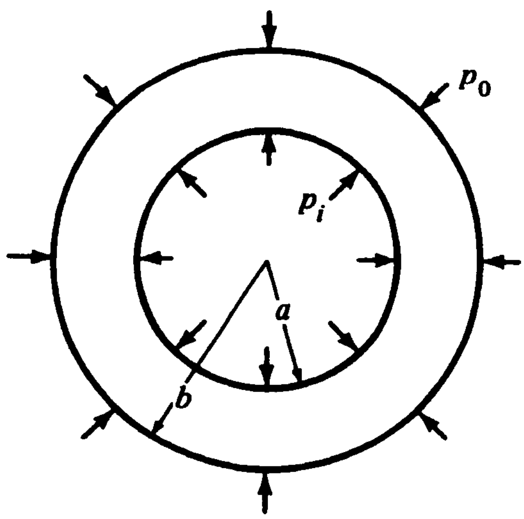


Figure E6-6.2.1

To investigate the variation of $(\sigma_r, \sigma_\theta)$ through the wall of the cylinder, consider the case $p_i = p$, $p_0 = 0$. Then Eqs. (b) and (d) yield

$$\begin{aligned} \sigma_r &= -\frac{a^2b^2p}{(b^2-a^2)r^2} + \frac{a^2p}{b^2-a^2} \\ \sigma_\theta &= \frac{a^2b^2p}{(b^2-a^2)r^2} + \frac{a^2p}{b^2-a^2} \end{aligned} \tag{e}$$

The change of $(\sigma_r, \sigma_\theta)$ with radial distance r is pictured in Fig. E6-6.2.2.

Example 6-6.3. Plane Strain Axisymmetrical Deformation of a Circular Cylinder. A thick-wall cylindrical pressure vessel with circular cross section undergoes linearly elastic deformation when subjected to a uniform external pressure acting on its outer lateral surface $r = b$. Its inner lateral surface at radius $r = a$ is constrained by a rigid cylindrical core so that its radial displacement $u = 0$ at $r = a$ (similar to Fig. P6-6.2, with $u = 0$). We wish to determine the stress components $(\sigma_r, \sigma_\theta, \sigma_z)$, where (r, θ) are polar coordinates in the cross section and z is the coordinate along the axis of the cylinder. The origin of coordinates (r, θ, z) is located at the center $(r = 0)$ of one of its end cross sections (where $z = 0$). We assume that the cylinder is free to expand laterally except at $r = a$ but is constrained axially so that a condition of plane strain relative to the (r, θ) plane exists.

Because the cylinder is loaded axisymmetrically, the theory of Section 6-6 applies. Thus, the stress components are independent of θ, and the tangential displacement component v (Fig. 6-1.1) is zero. Also, $\tau_{r\theta} = 0$ by Eqs. (6-6.2) and (6-6.4).

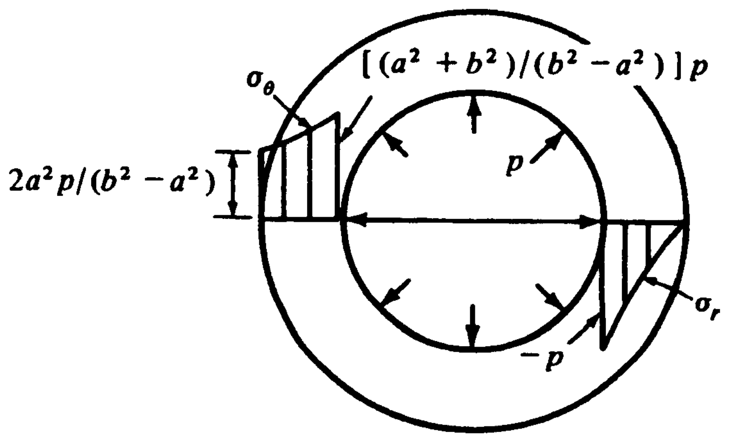


Figure E6-6.2.2

In the absence of body forces and temperature field, Eqs. (6-6.13) and (6-6.14) yield

$$\begin{aligned}\sigma_r &= \frac{C}{r^2} + \frac{A}{4}[2\log(r) - 1] + \frac{B}{2}\\ \sigma_\theta &= -\frac{C}{r^2} + \frac{A}{4}[2\log(r) + 1] + \frac{B}{2}\end{aligned} \tag{a}$$

Because the (r, θ) origin can be encircled by a contour entirely in the body even though the origin itself is not in the body (that is, it is located at $r = 0$), the constant $A = 0$. [See the discussion following Eq. (6-6.20).]

The strain–displacement relations for a linearly elastic isotropic medium for plane strain are [Eqs. (6-4.2) with $kT = 0$]

$$\begin{aligned}\epsilon_r &= \frac{du}{dr} = \frac{1-\nu^2}{E}[\sigma_r - \nu\sigma_\theta/(1-\nu)]\\ \epsilon_\theta &= \frac{u}{r} = \frac{1-\nu^2}{E}[\sigma_\theta - \nu\sigma_r/(1-\nu)]\end{aligned} \tag{b}$$

The second of Eqs. (b) yields, with Eqs. (a) and $A = 0$,

$$u = \frac{(1+\nu)r}{E}\left[-\frac{C}{r^2} + \frac{B(1-2\nu)}{2}\right] \tag{c}$$

The boundary condition $u = 0$ for $r = a$ yields with Eq. (c)

$$C = \frac{B(1-2\nu)a^2}{2} \tag{d}$$

The boundary condition $\sigma_r = -p$ for $r = b$ yields with Eqs. (a) and (d)

$$B = -\frac{2pb^2}{a^2(1-2\nu)+b^2}, \qquad C = -\frac{p(1-2\nu)a^2b^2}{a^2(1-2\nu)+b^2} \tag{e}$$

Equations (a) and (e) yield

$$\begin{aligned}\sigma_r &= -\frac{pb^2}{a^2(1-2\nu)+b^2}\left[1 + (1-2\nu)\frac{a^2}{r^2}\right]\\ \sigma_\theta &= -\frac{pb^2}{a^2(1-2\nu)+b^2}\left[1 - (1-2\nu)\frac{a^2}{r^2}\right]\end{aligned} \tag{f}$$

Therefore, because for plane strain $\sigma_z = \nu(\sigma_r + \sigma_\theta)$, we obtain by Eq. (f)

$$\sigma_z = -\frac{2\nu pb^2}{a^2(1-2\nu)+b^2} = \text{constant} \tag{g}$$

Equations (c) and (e) yield

$$u = -\frac{(1+\nu)(1-2\nu)pb^2r}{E[a^2(1-2\nu)+b^2]}\left[1-\frac{a^2}{r^2}\right] \tag{h}$$

Problem Set 6-6

1. Derive expressions for the radial and tangential components of displacement for the problem of Example 6-6.2.

2. A thin circular disk is given, which has outer radius b and inner radius a. The hole is expanded and a smooth, rigid plug of radius $a+\epsilon$ is inserted. Determine the stresses in the disk for this problem of generalized plane stress (Fig. P6-6.2).

3. A cylinder is cast of thermoplastic material in a steel mold (Fig. P6-6.3). The material solidifies at 210 °F. It is then cooled to room temperature, during which process the material "shrinks" (by thermal contraction) around the steel core. Estimate the maximum normal stress in the cylinder. The steel core has a 2-in. radius, and the plastic cylinder an original radius of 4 in. The coefficient of linear expansion is $k = 0.0002$ in./in./°F. $E = 10^5$ psi, $\nu = 0.5$.

4. Noting that the radial body force for a solid constant-thickness (thin) rotating disk is $R = \rho\omega^2 r$, where ρ is the mass density and ω is the angular frequency, show that a solution of the elasticity problem is given by $r\sigma_r = F$, $\sigma_\theta = (dF/dr) + \rho\omega^2 r^2$, where F satisfies the equation

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - F = -(3+\nu)\rho\omega^2 r^3 \tag{a}$$

Hence, show that the solution for F is

$$F = Ar + \frac{B}{r} - \frac{(3+\nu)\rho\omega^2 r^3}{8} \tag{b}$$

Derive expressions for the constants A and B for the solid disk (Fig. P6-6.4).

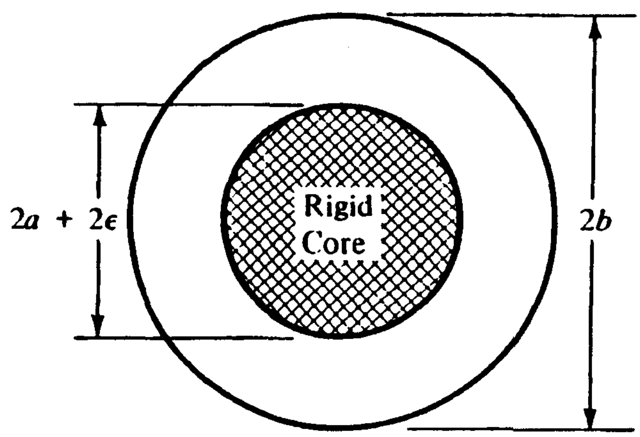


Figure P6-6.2

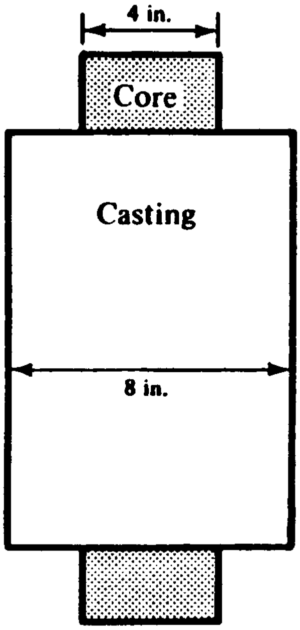


Figure P6-6.3

5. A steel disk with a hole 2 in. in diameter is shrunk on a shaft 2.003 in. in diameter. The disk has a constant thickness, and its outside diameter is 20 in. Assuming that the shaft is rigid, calculate the angular velocity at which the disk will become loose on the shaft (see Problem 4).

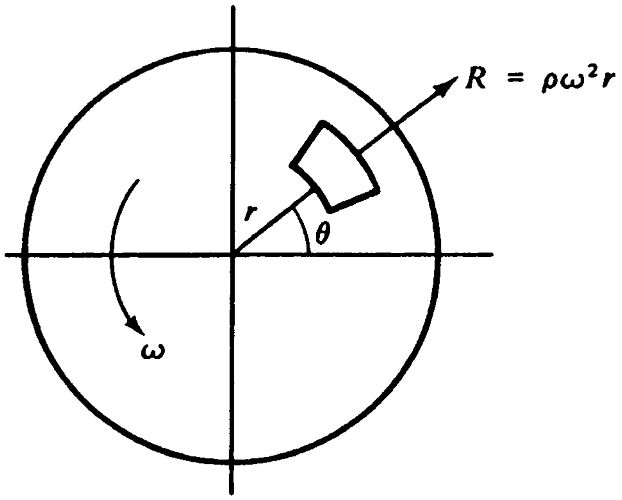


Figure P6-6.4

6. Consider the Airy stress function $F = Ar^2 \log r$, where (r, θ) are polar coordinates.
   (a) Compute the associated stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$.
   (b) Is the above Airy stress function a possible solution to a boundary-value problem of a complete ring $(a \le r \le b,\ 0 \le \theta \le 2\pi)$? Explain.
   (c) Is the above Airy stress function a possible solution to a boundary-value problem of a disk $(0 \le r \le b, 0 \le \theta \le 2\pi)$? Explain.
   (d) Is the above Airy stress function a possible solution to a boundary-value problem of an incomplete ring $(a \le r \le b,\ 0 \le \theta \le \theta_0 < 2\pi)$? Explain.
7. Plain strain axisymmetrical deformation of a circular cylinder: The cylinder has an inner radius a and an outer radius b. The inside of the cylinder is restrained such that the radial displacement is zero at $r = a$. The outside is subjected to a pressure p. Determine the stress σ_r, σ_θ, and σ_z as functions of r.
8. A thin circular disk has outer radius b and inner radius a (similar to Fig. P6-6.2). The radial displacement u at $r = a$ is zero. The outer boundary $(r = b)$ is subjected to pressure p and is otherwise unconstrained. Determine the stress components $(\sigma_r, \sigma_\theta)$ as functions of polar coordinate r. *Hint*: See Eqs. (6-6.4) and (6-6.5), and note that the boundary conditions must be satisfied. The problem is one of plane stress in the plane of the disk.
9. A thin circular annulus (inner radius$= a$; outer radius $= b$) is subjected to a temperature distribution T defined by the relation $kT = A(r^2 - a^2)$, where k and A are known constants. Derive expressions for the polar coordinate stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$. *Hint*: The compatibility equation for the axisymmetric plane stress problem is $d(r\epsilon_\theta)/dr = \epsilon_r$.
10. Let the disk of Problem 8 be subjected to pressure p at its inner surface $(r = a)$, which is now unconstrained. Let the outer radius be fixed so that the radial displacement $u = 0$. Determine $(\sigma_r, \sigma_\theta)$ as functions of r.
11. In addition to the constraints and load of Problem 8, let the disk be subjected to the temperature distribution T defined by $kT = A(r^2 - a^2)$, where k and A are known constants (see Problem 9). Determine the polar stress components $(\sigma_r, \sigma_\theta)$ as functions of r.

6-7 Plane-Elasticity Equations in Terms of Displacement Components

In this section we develop the plane-stress equilibrium equations for an isotropic homogeneous elastic material in the absence of temperature effects. In Section 6-9 we consider the plane stress problem of a variable-thickness disk of nonhomogeneous anisotropic material.

For plane stress, the stress–strain equilibrium equations in polar coordinates (r, θ), in the absence of temperature effects, are [see Eqs. (6-4.4)]

$$\begin{aligned}\sigma_r &= \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta)\\ \sigma_\theta &= \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r)\\ \tau_{r\theta} &= G\gamma_{r\theta} = \frac{E}{2(1+\nu)}\gamma_{r\theta}\end{aligned} \tag{6-7.1}$$

where $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ and $(\epsilon_r, \epsilon_\theta, \gamma_{r\theta})$ denote polar coordinate components of stress and strain, respectively, and where ν denotes Poisson's ratio, E Young's modulus, and G the shear modulus. The strain–displacement relations in plane polar coordinates are [Eq. 6-3.5)]

$$\begin{aligned} \epsilon_r &= u_r, \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r} v_\theta \\ \gamma_{r\theta} &= \frac{1}{r} u_\theta + v_r - \frac{v}{r} \end{aligned} \tag{6-7.2}$$

where (u, v) denote (r, θ) components of displacement, and where (r, θ) subscripts on (u, v) denote differentiation relative to (r, θ). Substitution of Eqs. (6-7.2) into Eqs. (6-7.1) yields

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(u_r + \nu\frac{u}{r} + \nu\frac{v_\theta}{r}\right) \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\frac{u}{r} + \frac{v_\theta}{r} + \nu u_r\right) \\ \tau_{r\theta} &= \frac{E}{2(1+\nu)}\left(\frac{u_\theta}{r} + v_r - \frac{v}{r}\right) \end{aligned} \tag{6-7.3}$$

The equilibrium equations are, with h = constant [see Eqs. (6-1.1)],

$$\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + B_r &= 0 \\ \frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} + B_\theta &= 0 \end{aligned} \tag{6-7.4}$$

Substitution of Eqs. (6-7.3) into Eqs. (6-7.4) yields

$$\begin{aligned} &\frac{E}{1-\nu^2}\left[u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} + \frac{(1+\nu)v_{r\theta}}{2r} - \frac{(3-\nu)v_\theta}{2r^2} + \frac{(1-\nu)u_{\theta\theta}}{2r^2}\right] + B_r = 0 \\ &\frac{E}{1-\nu^2}\left[\frac{(1+\nu)u_{r\theta}}{2r} + \frac{(3-\nu)u_\theta}{2r^2} + \frac{(1-\nu)v_{rr}}{2}\right. \\ &\qquad\qquad \left. + \frac{(1-\nu)v_r}{2r} - \frac{(1-\nu)v}{2r^2} + \frac{v_{\theta\theta}}{r^2}\right] + B_\theta = 0 \end{aligned} \tag{6-7.5}$$

In Eqs. (6-7.4) and (6-7.5) we have denoted body forces in the (r, θ) directions by (B_r, B_θ), respectively. Equations (6-7.5) are the equilibrium equations for plane stress problems in terms of displacement components (u, v) relative to polar coordinates (r, θ). They form the basis for study of plane stress boundary-value problems in polar coordinates. For the classical axisymmetric problem, $u = u(r)$, $v = 0$. Then Eqs. (6-7.5) reduce to the single equation

$$\frac{E}{1-\nu^2}\left[\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right] + B_r = 0 \tag{6-7.6}$$

For $B_r = 0$, Eq. (6-7.6) may be written

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = 0$$

and direct integration yields the solution

$$u = C_1 r + \frac{C_2}{r} \tag{6-7.7}$$

where the constants C_1, C_2 are determined by boundary conditions. For example, by Eqs. (6-7.3) and (6-7.7), we obtain, because $v = 0$,

$$\begin{aligned} \sigma_r &= \frac{EC_1}{1-\nu} - \frac{EC_2}{1+\nu}\frac{1}{r^2} \\ \sigma_\theta &= \frac{EC_1}{1-\nu} + \frac{EC_2}{1+\nu}\frac{1}{r^2} \end{aligned} \tag{6-7.8}$$

With the boundary conditions $\sigma_r = -p_0$ for $r = b$, $\sigma_r = -p_i$ for $r = a$, we obtain (see Example 6-6.2)

$$\begin{aligned} C_1 &= \frac{1-\nu}{E}\frac{p_i a^2 - p_0 b^2}{b^2 - a^2} \\ C_2 &= \frac{1+\nu}{E}\frac{a^2 b^2 (p_i - p_0)}{b^2 - a^2} \end{aligned} \tag{6-7.9}$$

Example 6-7.1. Stresses in a Rotating Disk Subjected to a Temperature Gradient. A thin solid disk of radius a rotates about an axis through its center $r = 0$ with a constant angular velocity ω. It is also subjected to a temperature field T defined by the relation $T = T_0 r/a$, where T_0 is a constant. We wish to determine the stresses in the disk and the increase of its diameter resulting from these effects.

The radial body force is $R = \rho r\omega^2$, and because ω = constant ($\alpha = 0$), the tangential body force $\Theta = 0$ (see Section 6-6). Hence by Eqs. (6-6.8) and (6-6.9), we have

$$\frac{1}{r^2}\frac{d}{dr}(r^2\sigma_r) = \frac{1}{r}(\sigma_r + \sigma_\theta) - \rho r\omega^2 \tag{a}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{r\theta}) = 0 \tag{b}$$

By Eq. (6-4.5), for plane stress we have (with $B_r = R = \rho r\omega^2$ and $B_\theta = \Theta = 0$)

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[\frac{d}{dr}(\sigma_r + \sigma_\theta + EkT) + (1+\nu)\rho r\omega^2\right]\right\} = 0 \tag{c}$$

Solving Eqs. (a) and (c) for $(\sigma_r, \sigma_\theta)$, with $T = T_0 r/a$, we find by the procedure used to obtain Eqs. (6-6.13) and (6-6.15), because $A = 0$,

$$\sigma_r = \frac{C}{r^2} + \frac{B}{2} - \frac{3+\nu}{8}\rho r^2\omega^2 - \frac{Ek}{3}T_0\frac{r}{a} \tag{d}$$

$$\sigma_\theta = -\frac{C}{r^2} + \frac{B}{2} - \frac{1+3\nu}{8}\rho r^2\omega^2 - \frac{2}{3}EkT_0\frac{r}{a} \tag{e}$$

Integration of Eq. (b) yields

$$\tau_{r\theta} = \frac{D}{r^2} \tag{f}$$

where D is a constant.

The boundary conditions at $r = a$ are $\sigma_r = 0$ and $\tau_{r\theta} = 0$. With these conditions, Eqs. (d) and (f) yield

$$\frac{C}{a^2} + \frac{B}{2} = \frac{3+\nu}{8}\rho a^2\omega^2 + \frac{Ek}{3}T_0 \tag{g}$$

$$D = 0 \tag{h}$$

At $r = 0$, $u = 0$. Hence, we must obtain an expression for u in terms of $(\sigma_r, \sigma_\theta)$ or constants C and B.

For plane stress, the strain–displacement relations are by the first two of Eqs. (6-4.4)

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + kT \tag{i}$$

$$\epsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + kT \tag{j}$$

Equation (j) yields

$$u = \frac{1}{E}[r\sigma_\theta - \nu r\sigma_r] + kTr \tag{k}$$

For $r = 0$, Eqs. (d), (e), and (k) yield

$$u = 0 = \frac{1}{E}\left[-\frac{C}{0} - \nu\frac{C}{0}\right] \tag{l}$$

Consequently, in order for u to be zero, $C = 0$. Hence Eq. (g) yields

$$B = \frac{3+v}{4}\rho a^2\omega^2 + \frac{2EkT_0}{3} \tag{m}$$

and Eqs. (d), (e), and (m) give

$$\sigma_r = \frac{3+v}{8}\rho\omega^2(a^2 - r^2) + \frac{EkT_0}{3}\left(1 - \frac{r}{a}\right) \tag{n}$$

$$\sigma_\theta = \frac{\rho\omega^2}{8}[(3+v)a^2 - (1+3v)r^2] + \frac{EkT_0}{3}\left(1 - \frac{2r}{a}\right) \tag{o}$$

By Eqs. (k), (n), and (o), the general expression for u is

$$u = \frac{(1-v)\rho\omega^2 r}{8E}[(3+v)a^2 - (1+v)r^2] + \frac{kT_0 r}{3}\left(1 + \frac{r}{a}\right) - \frac{vkT_0 r}{3}\left(1 - \frac{r}{a}\right) \tag{p}$$

Thus, for $r = a$,

$$u = \frac{(1-v)}{4}\rho a^3\omega^2 + \frac{2}{3}kT_0 a$$

and the increase in the diameter of the disk is

$$\Delta d = 2u = \frac{(1-v)}{2}\rho a^3\omega^2 + \frac{4}{3}kT_0 a \tag{q}$$

6-8 Plane Theory of Thermoelasticity

The plane theory of thermoelasticity is based on assumptions equivalent to those of plane-elasticity theory. Consequently, plane thermoelasticity consists of two cases: plane strain and plane stress (or, more generally, "generalized plane stress").

Plane Strain. We recall that a body is in a state of plane strain parallel to the (x, y) plane if the z displacement component w is constant and if (u, v), the (x, y) components of displacement, are functions of (x, y) only. Consequently, the strain–displacement relations in (x, y) coordinates reduce to

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = 0 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{xz} = \gamma_{yz} = 0 \end{aligned} \tag{6-8.1}$$

Substituting Eqs. (6-8.1) into the equations of thermoelasticity (see Section 4-12 in Chapter 4), we obtain relations for the plane strain theory of thermoelasticity.

In cylindrical coordinates (r, θ, z) the plane strain condition is expressed by the relations

$$u = u(r, \theta), \qquad v = v(r, \theta), \qquad w = \text{const}$$

Hence, the strain–displacement relations in cylindrical coordinates are [see Eqs. (2A-2.7) in Chapter 2 and (6-7.2)]

$$\begin{aligned} &\epsilon_r = \frac{\partial u}{\partial r}, \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \epsilon_z = 0 \\ &\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \qquad \gamma_{rz} = \gamma_{\theta z} = 0 \end{aligned} \tag{6-8.2}$$

The stress–strain–temperature relations in cylindrical coordinates are [see Eqs. (4-11.6) and (6-4.2)]

$$\begin{aligned} &\epsilon_r = E^{-1}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] + kT, \qquad \gamma_{r\theta} = G^{-1}\tau_{r\theta} \\ &\epsilon_\theta = E^{-1}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] + kT, \qquad \gamma_{rz} = \gamma_{\theta z} = 0 \\ &\epsilon_z = E^{-1}[\sigma_z - \nu(\sigma_r + \sigma_\theta)] + kT \end{aligned} \tag{6-8.3}$$

For plane strain $\epsilon_z = 0$; hence, the last of Eqs. (6-8.3) yields

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) - EkT \tag{6-8.4}$$

Substitution of Eq. (6-8.4) into Eq. (6-8.3) yields the stress–strain–temperature relations for plane strain:

$$\begin{aligned} &\epsilon_r = E^{-1}[(1 - \nu^2)\sigma_r - \nu(1 + \nu)\sigma_\theta] + (1 + \nu)kT \\ &\epsilon_\theta = E^{-1}[(1 - \nu^2)\sigma_\theta - \nu(1 + \nu)\sigma_r] + (1 + \nu)kT \\ &\gamma_{r\theta} = G^{-1}\tau_\theta \end{aligned} \tag{6-8.5}$$

For axisymmetric problems $v = 0$ and $\partial/\partial\theta = 0$, and Eqs. (6-8.2) are modified accordingly. Consequently, u and T are functions of r only.

For axially symmetric plane strain in the absence of body forces the equilibrium equations reduce to the single equation [see Eqs. (3A-2.7) and (2A-2.7) and let $\partial/\partial\theta = v = 0$]

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6-8.6}$$

Substituting Eqs. (6-8.2) into Eqs. (6-8.5), solving Eqs. (6-8.5) for $(\sigma_r, \sigma_\theta)$, and substituting the resulting equations into the equilibrium equation [Eq. (6-8.6)], we obtain

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \frac{1+v}{1-v}\frac{d(kT)}{dr}$$

Rewriting this equation, we obtain

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = \frac{1+v}{1-v}\frac{d(kT)}{dr} \tag{6-8.7}$$

Integration of Eq. (6-8.7) yields

$$u = \frac{1+v}{1-v}\frac{1}{r}\int_a^r \rho kT \, d\rho + Ar + \frac{B}{r} \tag{6-8.8}$$

Equations (6-8.7) to (6-8.8) and corresponding modifications of the equations of Section 4-12 in Chapter 4 summarize the plane strain theory of thermoelasticity.

Plane Stress. A body is in a state of plane stress in the (x, y) plane if $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. Substitution of these conditions into the general thermoelasticity theory of Section 4-12 in Chapter 4 yields the corresponding equations of plane stress thermoelasticity.

In cylindrical coordinates, the stress–strain–temperature relations for plane stress are [see Eqs. (6-8.3)]

$$\begin{aligned} \epsilon_r &= E^{-1}(\sigma_r - v\sigma_\theta) + kT \\ \epsilon_\theta &= E^{-1}(\sigma_\theta - v\sigma_r) + kT \\ \epsilon_z &= -\frac{v}{E}(\sigma_r + \sigma_\theta) + kT \end{aligned} \tag{6-8.9}$$

Inverting the first two of Eq. (6-8.9), we obtain

$$\begin{aligned} \sigma_r &= \frac{E}{1-v^2}(\epsilon_r + v\epsilon_\theta) - \frac{EkT}{1-v} \\ \sigma_\theta &= \frac{E}{1-v^2}(\epsilon_\theta + v\epsilon_r) - \frac{EkT}{1-v} \end{aligned} \tag{6-8.10}$$

Substitution of Eqs. (6-8.10) into the last of Eqs. (6-8.9) yields

$$\epsilon_z = \frac{-v}{1-v}(\epsilon_r + \epsilon_\theta) + \frac{1+v}{1-v}kT \tag{6-8.11}$$

Equation (6-8.6) is the equilibrium condition for axially symmetric plane stress thermoelasticity, as $\sigma_z = \tau_{rz} = \tau_{\theta z} = 0$. Also, because $v = \partial/\partial\theta = 0$ for axial symmetry, the strain–displacement relations [Eqs. (6-8.2)] reduce to

$$\epsilon_r = \frac{du}{dr}, \qquad \epsilon_\theta = \frac{u}{r} \tag{6-8.12}$$

where u is the displacement in the r direction.

Substitution of Eqs. (6-8.12) and (6-8.10) into Eqs. (6-8.6) yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1+\nu)\frac{d(kT)}{dr} \tag{6-8.13}$$

Integration of Eq. (6-8.13) yields

$$u = (1+\nu)\frac{1}{r}\int_a^r kT\rho\, d\rho + Ar + \frac{B}{r} \tag{6-8.14}$$

Equations (6-8.9) to (6-8.14) and corresponding modifications of the equations of Section 4-12 summarize the theory of plane stress thermoelasticity. Plane stress thermoelasticity problems of radial heating of a thin circular disk and axial heating of beams and strips are important in practice.

Problem Set 6-8

1. (a) Show that $u = \sum_{n=0}^{\infty} a_n \cos n\theta$, $v = \sum_{n=0}^{\infty} b_n \sin n\theta$, where (u, v) denote polar coordinates (r, θ) components of displacement and the coefficients a_n, b_n are functions of r only, is a possible solution of the plane stress equations of equilibrium expressed in terms of displacement components (u, v).
   (b) Derive the differential equations that define the coefficients a_n, b_n.
2. Assume that the Airy stress function F is of the form $F = f(\theta)$, where $f(\theta)$ is a function of θ, the polar coordinate angle of polar coordinates (r, θ).
   (a) Derive the explicit form for $f(\theta)$ for the case of the plane stress problem of a ring, in the region $a \leq r \leq b$, under uniform shearing stresses applied at the inner ($r = a$) and outer ($r = b$) surfaces of the ring. Neglect body forces and inertia forces.
   (b) Derive explicit expressions for displacement components (u, v) relative to polar coordinates (r, θ), respectively, expressing the results in terms of the applied stresses and the radii a and b.
3. A thin circular disk of radius a is subjected to a temperature distribution

$$T = T_0\left(1 - \frac{r}{a}\right)$$

where T_0 is a known constant. The compatibility equation for axisymmetrical polar coordinate problems with thermal effects is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = -EkT + C$$

where k is the coefficient of thermal expansion and C is an unknown constant of integration (to be determined by boundary conditions). Determine the change in diameter of the disk due to the applied temperature.

4. Derive the compatibility equation given in Problem 3.

5. A long mine tunnel of radius a is cut in deep rock. Before the tunnel is cut, the rock is subjected to uniform pressure p. Considering the rock to be an infinite, homogeneous elastic medium with elastic constants E and ν, determine the inward radial displacement at the surface of the tunnel due to the excavation.

6. A circular annular disk rotates with constant angular velocity ω about the axis O, perpendicular to the plane of the disk (Fig. P6-8.6). The inner radius of the disk is located at $r = a$, the outer radius at $r = b$.The inside radius is restrained to prevent radial displacement. Assume that a state of plane stress relative to the plane of the disk exists.
   (a) Derive the equations of motion of the disk in terms of displacement components relative to polar coordinates (r, θ).
   (b) Integrate the equations to determine the radial displacement u.
   (c) Determine the constants of integrations.

7. Modify the equations of Section 4-12 in Chapter 4 for plane strain. Repeat for plane stress.

8. Let $T = T(r, \theta)$ for a plane thermoelasticity problem in polar coordinates (r, θ). Determine an explicit expression for $T(r, \theta)$ for the steady-state case in absence of

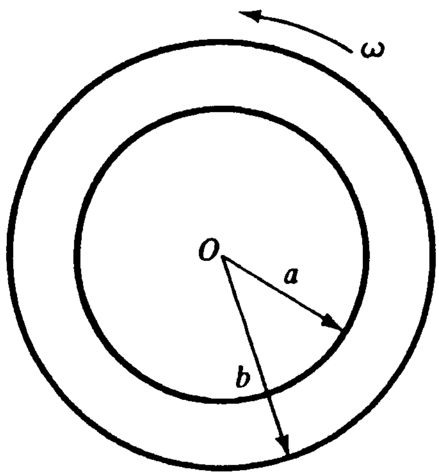


Figure P6-8.6

heat source, expressing $T(r, \theta)$ in the form $T = T_1(r) + T_2(r, \theta)$, where $T_1(r)$ is the part of $T(r, \theta)$ dependent upon r alone. That is, show that

$$T_1(r) = A_0 + B_0 \log r$$
$$T_2(r, \theta) = \sum_{n=1}^{\infty} [(A_n r^n + B_n r^{-n}) \cos n\theta + (c_n r^n + D_n r^{-n}) \sin n\theta]$$

where A_n, B_n, C_n, D_n are constants.

9. In Problem 8 set all constants except B_1 and D_1 equal to zero. For the resulting temperature field, determine the stress produced in a hollow circular cylinder defined by cylindrical coordinates (r, θ, z), the z axis coinciding with the longitudinal axis of the cyliner. Assume that axial displacement of the cylinder is prevented.

10. A nuclear fuel element in the form of a solid right-circular cylinder is free to expand laterally but not axially. It is subjected to a radiation heat source in the form of the Gaussian distribution

$$Q = Ae^{-\alpha^2 r^2}$$

where α^2 is a constant and r is the radial coordinate. Generally, $\alpha^2 \ll 1$. Compute the temperature distribution T. What reasonable approximation may be used for T? Determine the stress distribution in the cylinder. What practical restriction must be imposed on A?

11. A solid plane circular disk of radius a is subjected to the temperature distribution T given by $kT = A + Br \cos \theta + Cr \sin \theta$, where A, B, C are constants and (r, θ) are polar coordinates with origin at the center of the disk. The disk is not restrained at its boundary $r = a$.
    (a) Show that the solution of the plane stress problem of the disk is $\sigma_r = \sigma_\theta = \tau_{r\theta} = 0$.
    (b) Derive explicit expressions for the radial and tangential displacement components (u, v), respectively.
    (c) Write the boundary conditions that determine the arbitrary constants of integration of part (b).

6-9 Disk of Variable Thickness and Nonhomogeneous Anisotropic Material

In this section we treat the variable-thickness elastic disk made of nonhomogeneous anisotropic material relative to polar coordinates (r, θ). We assume that the stress components and body forces are functions of radial distance r from the center of the disk.

The equilibrium equations are [Eqs. (6-1.1)]

$$\begin{aligned}\frac{d}{dr}(h\sigma_r) + \frac{h}{r}(\sigma_r - \sigma_\theta) + hB_r = 0 \\ \frac{d}{dr}(h\tau_{r\theta}) + \frac{2h}{r}\tau_{r\theta} + hB_\theta = 0\end{aligned} \tag{6-9.1}$$

where $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ denote stress components relative (r, θ) coordinates, $h = h(r)$ denotes the disk thickness, and (B_r, B_θ) denote the body forces per unit volume in the (r, θ) directions, respectively.

For the material being considered, the stress–strain–temperature relations are

$$\begin{aligned}\sigma_r &= C_{11}\epsilon_r + C_{12}\epsilon_\theta + C_{13}\gamma_{r\theta} - C_1 T \\ \sigma_\theta &= C_{12}\epsilon_r + C_{22}\epsilon_\theta + C_{23}\gamma_{r\theta} - C_2 T \\ \tau_{r\theta} &= C_{13}\epsilon_r + C_{23}\epsilon_\theta + C_{33}\gamma_{r\theta} - C_3 T\end{aligned} \tag{6-9.2}$$

where $C_{ij} = C_{ji} = C_{ij}(r)$ are elastic constants, $C_i = C_i(r)$ are thermoelastic constants, $T = T(r)$ denotes temperature, and $(\epsilon_r, \epsilon_\theta, \gamma_{r\theta})$ are strain components.

Inverting Eqs. (6-9.2), we obtain

$$\begin{aligned}\epsilon_r &= S_{11}\sigma_r + S_{12}\sigma_\theta + S_{13}\tau_{r\theta} + k_1 T \\ \epsilon_\theta &= S_{12}\sigma_r + S_{22}\sigma_\theta + S_{23}\tau_{r\theta} + k_2 T \\ \gamma_{r\theta} &= S_{13}\sigma_r + S_{23}\sigma_\theta + S_{33}\tau_{r\theta} + k_3 T\end{aligned} \tag{6-9.3}$$

where (k_1, k_2, k_3) are linear thermal expansion coefficients related to C_i and C_{ij} by the relations

$$\begin{aligned}Ck_1 &= C_1\begin{vmatrix} C_{22} & C_{23} \\ C_{23} & C_{33}\end{vmatrix} - C_2\begin{vmatrix} C_{12} & C_{13} \\ C_{23} & C_{33}\end{vmatrix} + C_3\begin{vmatrix} C_{12} & C_{13} \\ C_{22} & C_{23}\end{vmatrix} \\ Ck_2 &= -C_1\begin{vmatrix} C_{12} & C_{23} \\ C_{13} & C_{33}\end{vmatrix} + C_2\begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33}\end{vmatrix} - C_3\begin{vmatrix} C_{11} & C_{13} \\ C_{12} & C_{23}\end{vmatrix} \\ Ck_3 &= C_1\begin{vmatrix} C_{12} & C_{22} \\ C_{13} & C_{23}\end{vmatrix} - C_2\begin{vmatrix} C_{11} & C_{12} \\ C_{13} & C_{23}\end{vmatrix} + C_3\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22}\end{vmatrix}\end{aligned} \tag{6-9.4}$$

and

$$
\begin{aligned}
S_{11} &= \frac{1}{C}\begin{vmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{vmatrix}, \qquad S_{12} = S_{21} = -\frac{1}{C}\begin{vmatrix} C_{12} & C_{13} \\ C_{23} & C_{33} \end{vmatrix} \\
S_{13} = S_{31} &= \frac{1}{C}\begin{vmatrix} C_{12} & C_{13} \\ C_{22} & C_{23} \end{vmatrix}, \qquad S_{22} = \frac{1}{C}\begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} \\
S_{23} = S_{32} &= -\frac{1}{C}\begin{vmatrix} C_{11} & C_{13} \\ C_{12} & C_{23} \end{vmatrix}, \qquad S_{33} = \frac{1}{C}\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} \\
C &= \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}
\end{aligned}
\tag{6-9.5}
$$

For the type of problem considered here, $u = u(r)$ and $v = v(r)$. Then Eqs. (6-3.5) reduce to

$$
\epsilon_r = u', \qquad \epsilon_\theta = \frac{u}{r}, \qquad \gamma_{r\theta} = v' - \frac{v}{r} \tag{6-9.6}
$$

where primes denote derivatives with respect to r.

Equations (6-9.1), (6-9.2), and (6-9.6) yield

$$
\begin{aligned}
\sigma_r &= C_{11}u' + C_{12}\frac{u}{r} + C_{13}\left(v' - \frac{v}{r}\right) - C_1 T \\
\sigma_\theta &= C_{12}u' + C_{22}\frac{u}{r} + C_{23}\left(v' - \frac{v}{r}\right) - C_2 T \\
\tau_{r\theta} &= C_{13}u' + C_{23}\frac{u}{r} + C_{33}\left(v' - \frac{v}{r}\right) - C_3 T
\end{aligned}
\tag{6-9.7}
$$

and

$$
\begin{aligned}
u'' + R_1 u' + R_2 u + R_3 v'' + R_4 v' + R_5 v &= R_6 \\
v'' + P_1 v' + P_2 v + P_3 u'' + P_4 u' + P_5 u &= P_6
\end{aligned}
\tag{6-9.8}
$$

where

$$
\begin{aligned}
rR_1 &= 1 + \frac{r}{\bar{C}_{11}}\bar{C}'_{11} \\
r^2R_2 &= \frac{r}{\bar{C}_{11}}\bar{C}'_{12} - \frac{\bar{C}_{22}}{\bar{C}_{11}} \\
R_3 &= \frac{\bar{C}_{13}}{\bar{C}_{11}} \\
rR_4 &= \frac{r}{\bar{C}_{11}}\bar{C}'_{13} - \frac{\bar{C}_{23}}{\bar{C}_{11}} \\
rR_5 &= -R_4 \\
R_6 &= \frac{1}{\bar{C}_{11}}\left[-\bar{B}_r + \bar{C}_1 T' + \left(\bar{C}'_1 + \frac{\bar{C}_1}{r} - \frac{\bar{C}_2}{r}\right)T\right] \\
rP_1 &= 1 + \frac{r}{\bar{C}_{33}}\bar{C}'_{33} \\
rP_2 &= -P_1 \\
P_3 &= \frac{\bar{C}_{13}}{\bar{C}_{33}} \\
rP_4 &= \frac{2\bar{C}_{13} + \bar{C}_{23}}{\bar{C}_{33}} + \frac{r}{\bar{C}_3}\bar{C}'_{13} \\
r^2P_5 &= \frac{\bar{C}_{23}}{\bar{C}_{33}} + \frac{r}{\bar{C}_{33}}\bar{C}'_{23} \\
P_6 &= \frac{1}{\bar{C}_{33}}\left[-\bar{B}_\theta + \left(\frac{2\bar{C}_3 + r\bar{C}'_3}{r}\right)T + \bar{C}_3 T'\right]
\end{aligned}
\tag{6-9.9}
$$

where

$$\bar{C}_{ij} = hC_{ij}, \qquad \bar{C}_i = hC_i, \qquad \bar{B}_r = hB_r, \qquad \bar{B}_\theta = hB_\theta \tag{6-9.10}$$

If the disk rotates with angular velocity ω and angular acceleration α,

$$B_r = \rho\omega^2, \qquad B_\theta = -\rho\alpha r \tag{6-9.11}$$

where ρ denotes mass per unit volume.

Boundary Conditions. We consider two cases.

Case 1.

$$\begin{aligned} &\text{For } r = a, \quad \sigma_r = \sigma_a, \quad \tau_{r\theta} = \tau_a \\ &\text{For } r = b, \quad \sigma_r = \sigma_b, \quad \tau_{r\theta} = \tau_b \end{aligned} \tag{6-9.12}$$

where $\sigma_a, \sigma_b, \tau_a, \tau_b$ are prescribed constants.

Case 2.

$$\begin{aligned} &\text{For } r = a, \quad u = u_a, \quad v = v_a \\ &\text{For } r = b, \quad \sigma_r = \sigma_b, \quad \tau_{r\theta} = \tau_b \end{aligned} \tag{6-9.13}$$

where $\sigma_b, \tau_b, u_a, v_a$ are presribed constants.

Substitution of Eqs. (6-9.7) into Eqs. (6-9.12) and (6-9.13) yields the boundary conditions in terms of (u, v) in the form

Case 1. For $r = a$,

$$\begin{aligned} C_{11}u' + C_{12}\frac{u}{a} + C_{13}\left(v' - \frac{v}{a}\right) - C_1 T_a = \sigma_a \\ C_{13}u' + C_{23}\frac{u}{a} + C_{33}\left(v' - \frac{v}{a}\right) - C_3 T_a = \tau_a \end{aligned} \tag{6-9.14}$$

where $T_a = T$ evaluated at $r = a$. For $r = b$,

$$\begin{aligned} C_{11}u' + C_{12}\frac{u}{b} + C_{13}\left(v' - \frac{v}{b}\right) - C_1 T_b = \sigma_b \\ C_{13}u' + C_{23}\frac{u}{b} + C_{33}\left(v' - \frac{v}{b}\right) - C_3 T_b = \tau_b \end{aligned} \tag{6-9.15}$$

where $T_b = T$ evaluated at $r = b$.

Case 2. For $r = a$.

$$u = u_a, \qquad v = v_a \tag{6-9.16}$$

For $r = b$,

$$\begin{aligned} C_{11}u' + C_{12}\frac{u}{b} + C_{13}\left(v' - \frac{v}{b}\right) - C_1 T_b = \sigma_b \\ C_{13}u' + C_{23}\frac{u}{b} + C_{33}\left(v' - \frac{v}{b}\right) - C_3 T_b = \tau_b \end{aligned} \tag{6-9.17}$$

Equations (6-9.8) with appropriate boundary conditions [Eqs. (6-9.14) and (6-9.15) or Eqs. (6-9.16) and (6-9.17)] define the described disk problem for plane stress with nonhomogeneous anisotropic material. If $C_{13} = C_{23} = 0$ and $v = 0$, the above theory reduces to the axisymmetric plane stress problem of the orthotropic disk. If $C_{13} = C_{23} = 0$ and $v \neq 0$, the above theory uncouples into two problems, one that defines u and the other that defines v. The defining equations for the u problem are the first of Eqs. (6-9.8) with $R_3 = R_4 = R_5 = 0$ and the boundary conditions $\sigma_r = \sigma_a$ for $r = a$, $\sigma_r = \sigma_b$ for $r = b$ [Case 1, Eq. (6-9.12)], or $u = u_a$ for $r = a$, $\sigma_r = \sigma_b$ for $r = b$ [Case 2, Eq. (6-9.13)]. The defining equations for v are the second of Eqs. (6-9.8) with $P_3 = P_4 = P_5 = 0$ and the boundary conditions $\tau_{r\theta} = \tau_a$ for $r = a$, $\tau_{r\theta} = \tau_b$ for $r = b$ [Case 1, Eq. (6-9.12)], or $v = v_a$ for $r = a$, $\tau_{r\theta} = \tau_b$ for $r = b$ [Case 2, Eq. (6-9.13)]. The finite difference method may be used to solve the boundary-value problem described above.

Problem Set 6-9

1. An annular plane region R defined by $a \leq r \leq b$ is subjected to uniform pressure p_a at $r = a$ and p_b at $r = b$. The stress–strain relations of the material relative to polar coordinates (r, θ) are

$$\sigma_r = C_{11}\epsilon_r + C_{12}\epsilon_\theta$$
$$\sigma_\theta = C_{12}\epsilon_r + C_{22}\epsilon_\theta$$
$$\tau_{r\theta} = C_{33}\gamma_{r\theta}$$

where C_{ij} are constant elastic coefficients.

(a) Considering the physical nature of the problem, express the equilibrium equations in terms of (u, v), the (r, θ) displacement components.

(b) Show that the radial displacement component u is of the form

$$u = Ar^{-n} + Br^{n}$$

where n is an explicit function of the elastic constants C_{ij} and (A, B) are constants.

(c) Write the conditions that define the constants A and B.

6-10 Stress Concentration Problem of Circular Hole in Plate

The general solution for the Airy stress function [Eq. (6-5.5)] includes a number of special cases of importance (Section 6-11). In this article we single out the particularly important problem of a plane rectangular region with interior circular hole and subjected to uniformly distributed edge stresses (Kirsch, 1898). As a special case of this problem, we treat in detail the case of uniformly distributed normal stress along two opposite edges (Fig. 6-10.1). More generally, normal stress and shear stress may be distributed uniformly along all edges. We assume that the hole is sufficiently small compared to typical overall dimensions of the region so that

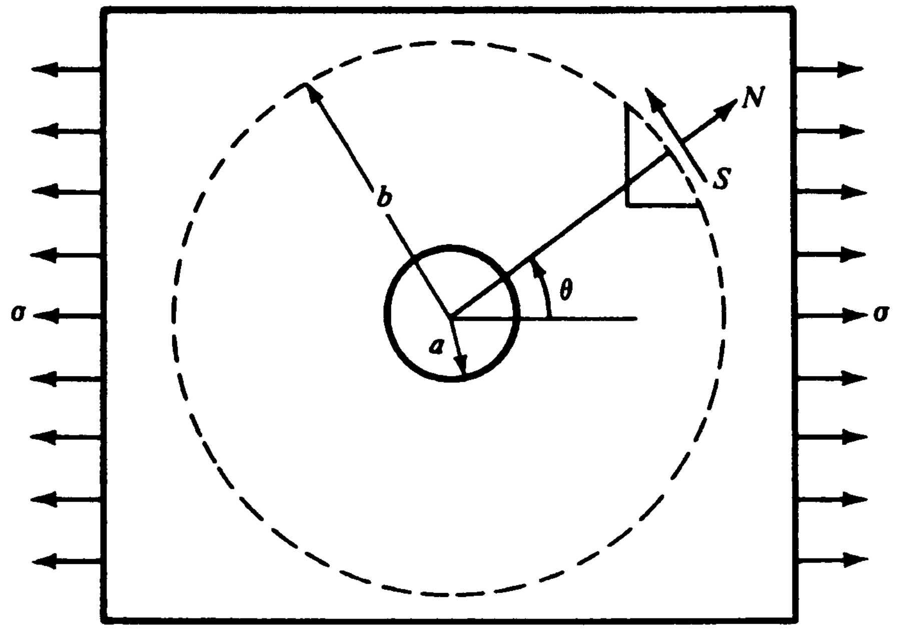


Figure 6-10.1

there exists regions far removed from the hole in which the stresses are essentially unaffected by the hole. Hence, for a circle $r = b$ scribed in the region ($b \gg a$), the stress distribution is obtained by considering the equilibrium state of an element (Fig. 6-10.2). Thus, we find

$$\begin{aligned} N &= \sigma \cos^2 \theta = \frac{\sigma}{2}(1 + \cos 2\theta) \\ S &= -\sigma \sin \theta \cos \theta = -\frac{\sigma}{2} \sin 2\theta \end{aligned} \tag{6-10.1}$$

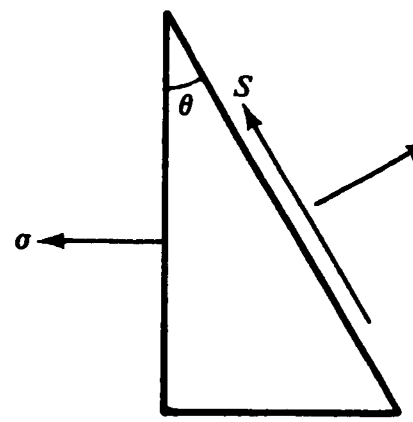


Figure 6-10.2

For simplicity, we may consider the stress components [Eq. (6-10.1)] as the sum of two stress states (Fig. 6-10.3)

$$\begin{aligned} N &= N_1 + N_2 \\ S &= S_1 + S_2 \end{aligned} \tag{6-10.2}$$

where

$$N_1 = \frac{\sigma}{2}, \qquad S_1 = 0 \tag{6-10.3}$$

and

$$N_1 = \frac{\sigma}{2}\cos 2\theta, \qquad S_2 = -\frac{\sigma}{2}\sin 2\theta \tag{6-10.4}$$

The stress distribution for state 1 is described by the results of Example (6-6.2), with $p_i = 0$, $p_0 = -\sigma/2$, and $a \ll b$. The stress distribution of state 2 may be described by the Airy stress function

$$F(r, \theta) = f(r)\cos 2\theta \tag{6-10.5}$$

Substitution of Eqs. (6-10.5) into the compatibility equation $\nabla^2\nabla^2 F = 0$ yields the equation for $f(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0$$

the solution of which is

$$f(r) = Ar^2 + Br^4 + C\frac{1}{r^2} + D \tag{6-10.6}$$

Hence,

$$F(r, \theta) = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos 2\theta \tag{6-10.7}$$

Equation (6-10.7) corresponds to the term in the first summation of Eq. (6-5.5), with $n = 2$, where A, B, C, D are constants to be determined by the boundary conditions for state 2 (Fig. 6-10.3).

The stress components for state 2 are, by Eqs. (6-10.7) and (6-2.2),

$$\begin{aligned} \sigma_r &= -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta \\ \sigma_\theta &= \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} &= \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta \end{aligned} \tag{6-10.8}$$

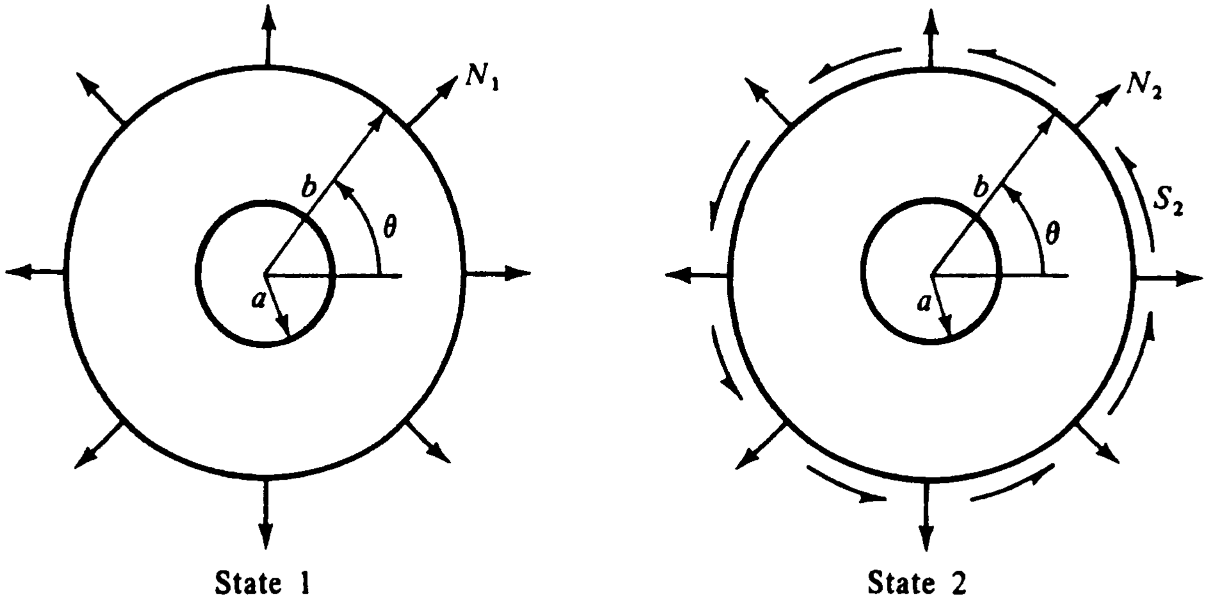


Figure 6-10.3

Accordingly, with Eqs. (6-10.8) and the boundary conditions

$$\begin{aligned}&\sigma_r = \tau_{r\theta} = 0 \qquad \text{for} \qquad r = a\\&\sigma_r = \tfrac{1}{2}\sigma\cos 2\theta, \qquad \tau_{r\theta} = -\tfrac{1}{2}\sigma\sin 2\theta \qquad \text{for} \qquad r = b\end{aligned} \tag{6-10.9}$$

the values of A, B, C, D are

$$A = -\frac{\sigma}{4}, \qquad B = 0, \qquad C = -\frac{a^4}{4}\sigma, \qquad D = \frac{a^2}{2}\sigma \tag{6-10.10}$$

Then superposition of Eqs. (6-10.8) and Eqs. (b) of Example 6-6.2 [with Eqs. (d) under the conditions that $p_i = 0$, $p_0 = -(\sigma/2)$ and $b \gg a$] yields the stress state in the plane region with small circular hole (Fig. 6-10.1):

$$\begin{aligned}\sigma_r &= \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta\\\sigma_\theta &= \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\\\tau_{r\theta} &= -\frac{\sigma}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta\end{aligned} \tag{6-10.11}$$

We note that as $r \to b\ (\gg a)$, the stress state given by Eqs. (6-10.11) satisfies the conditions for $r = b$ [Eqs. (6-10.1)]. Also for $r = a$,

$$\sigma_r = \tau_{r\theta} = 0, \qquad \sigma_\theta = \sigma(1 - 2\cos 2\theta)$$

For $\theta = \pi/2$, $3\pi/2$, σ_θ attains its maximum value of $(\sigma_\theta)_{\max} = 3\sigma$. [In general, $(\sigma_\theta)_{\max} = k\sigma$, where k is called the *stress concentration factor*.] For $\theta = 0$, π, σ_θ attains a compressive value of $-\sigma$. Thus, σ_θ attains a maximum tensile value of three times the uniformly distributed stress σ, at the hole $r = a$, for $\theta = \pi/2$, $3\pi/2$ (Fig. 6-10.4).

Because for $\theta = \pi/2$, $3\pi/2$, $\sigma_\theta = (\sigma/2)(2 + a^2/r^2 + 3a^4/r^4)$, $\sigma_\theta \to \sigma$ rapidly as r increases. Hence, the effect of the hole is of local character, the hole producing a *stress concentration* effect that increases the maximum stress several fold in the vicinity of the hole over the nominal stress value σ.

By superposition, we may also show $(\sigma_\theta)_{\max} = 2\sigma$ everywhere at the boundary of the hole, when uniform tensile stress σ is applied along all straight edges of the plate. Furthermore, if a uniform compressive stress of magnitude σ is applied to two opposite edges (say, the horizontal edges in Fig. 6-10.1) and a uniform tensile stress σ is applied simultaneously to the other edges (Fig. 6-10.1), at the hole, then

$$\begin{aligned}&\sigma_\theta = 4\sigma \qquad \text{for} \qquad \theta = \pi/2, 3\pi/2\\&\sigma_\theta = -4\sigma \qquad \text{for} \qquad \theta = 0, \pi\end{aligned}$$

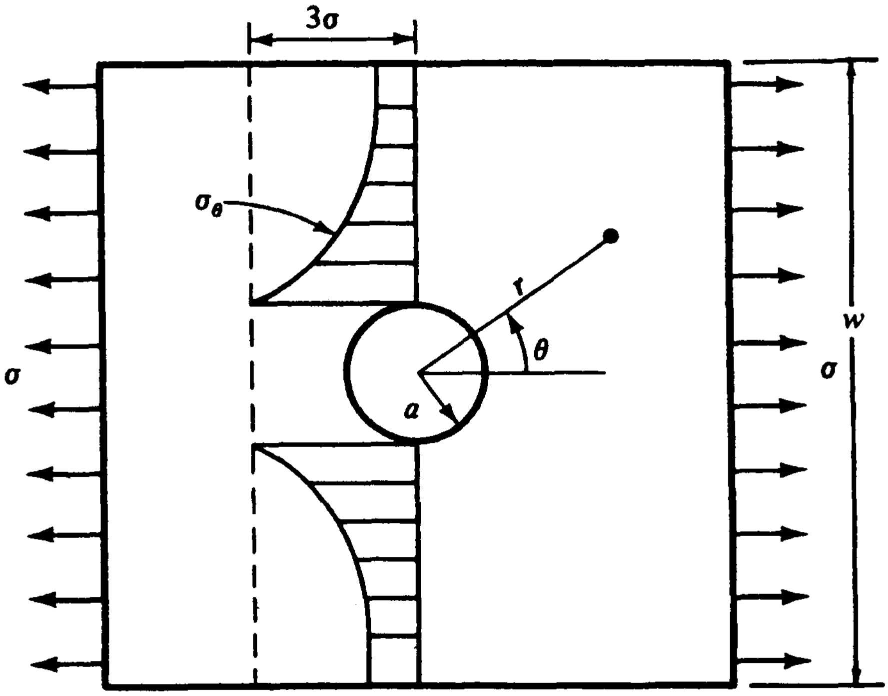


Figure 6-10.4

The large stress concentration effect that occurs at small holes in structural elements is of considerable importance to the designer. Much effort is expended to determine these effects and to design elements that minimize such effects (Savin, 1961).[2]

The displacement components in the region may be determined by the method noted in Section 6-6, that is, by direct integration of the strain–displacement relations [Eqs. (6-3.5)].

Plane Strain under General Loading. More generally, the stress concentration problem of a circular hole in a plate subject to boundary stresses σ_x, σ_y, τ_{xy} may be solved by the method of superposition. For example, consider (x, y) axes with origin at the center of the hole, with the x-axis in the horizontal direction and the y-axis in the vertical direction (Figs. 6-10.1 and 6-10.4). On distant boundary planes perpendicular to the x-axis, stresses σ_x, τ_{xy} act, and on distant boundary planes perpendicular to the y-axis, stresses σ_y, τ_{xy} act. The stress components σ_x, σ_y, τ_{xy} are assumed to act in the positive sense (see Fig. 3-2.2 in Chapter 3).

[2] Savin's book is devoted entirely to methods of calculating stress concentration factors.

The stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ (see Fig. 6-1.1) at a point (r, θ), as in Fig. 6-10.4, are

$$\begin{aligned}
\sigma_r &= \left(\frac{\sigma_x + \sigma_y}{2}\right)\left(1 - \frac{a^2}{r^2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\left(1 + \frac{3a^4}{r^4} - 4\frac{a^2}{r^2}\right)\cos 2\theta \\
&\quad + \tau_{xy}\left(1 + \frac{3a^4}{r^4} - 4\frac{a^2}{r^2}\right)\sin 2\theta \\
\sigma_\theta &= \left(\frac{\sigma_x + \sigma_y}{2}\right)\left(1 + \frac{a^2}{r^2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\
&\quad - \tau_{xy}\left(1 + \frac{3a^4}{r^4}\right)\sin 2\theta \\
\tau_{r\theta} &= -\left(\frac{\sigma_x - \sigma_y}{2}\right)\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta + \tau_{xy}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\cos 2\theta
\end{aligned} \tag{6-10.12}$$

For plane strain, $\epsilon_z = 0$. Hence,

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \tag{6-10.13}$$

The first two of Eqs. (6-10.12) and Eq. (6-10.13) yield

$$\sigma_z = \nu\left[\sigma_x + \sigma_y - 2(\sigma_x - \sigma_y)\frac{a^2}{r^2}\cos 2\theta - \frac{4a^2}{r^2}\tau_{xy}\sin 2\theta\right] \tag{6-10.14}$$

For $r = a$, Eqs. (6-10.12) and (6-10.14) yield

$$\begin{aligned}
&\sigma_r = 0, \qquad \tau_{r\theta} = 0 \\
&\sigma_\theta = \sigma_x + \sigma_y - 2(\sigma_x - \sigma_y)\cos 2\theta - 4\tau_{xy}\sin 2\theta \\
&\sigma_z = \nu\sigma_\theta
\end{aligned} \tag{6-10.15}$$

The maximum, minimum values of σ_θ, hence σ_z, are given by the condition

$$\tan 2\theta = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \tag{6-10.16}$$

For example, for $\tau_{xy} = 0$, Eq. (6-10.16) yields $\tan 2\theta = 0$, or $\theta = 0$ (or π), $\pi/2$ (or $3\pi/2$). Thus, by Eq. (6-10.15) we obtain

$$\sigma_\theta = -\sigma_x + 3\sigma_y, \qquad \text{for } \theta = 0, \pi; \quad r = a \tag{6-10.17}$$

and

$$\sigma_\theta = 3\sigma_x - \sigma_y, \qquad \text{for } \theta = \pi/2, 3\pi/2; \quad r = a \tag{6-10.18}$$

For the cases ($\sigma_x = \sigma$, $\sigma_y = 0$), ($\sigma_x = \sigma_y = \sigma$] and ($\sigma_x = -\sigma_y = \sigma$), Eqs. (6-10.17) and (6-10.18) yield the results obtained in the discussion following Eqs. (6-10.12). For $\sigma_x = \sigma_y = 0$, $\sigma_\theta = -4\tau_{xy} \sin 2\theta$. Hence,

$$\begin{aligned} (\sigma_\theta)_{\max} &= 4\tau_{xy}, \qquad \text{for } \theta = 3\pi/4; \quad r = a \\ (\sigma_\theta)_{\min} &= -4\tau_{xy}, \qquad \text{for } \theta = \pi/4; \quad r = a \end{aligned} \tag{6-10.19}$$

Applications of Eqs. (6-10.12) to rock-mechanics problems have been given by Leeman and Hayes (1966) and to deep mine-shaft problems by Chan and Beus (1985).

Large Holes. Equations (6-10.11) and (6-10.12) are applicable for the condition $a \ll b$; that is, for small circular holes relative to the loaded regions (Fig. 6-10.1). For a large hole (the radius of the hole being large compared to the smallest dimension of the region, which is the lateral width w in Fig. 6-10.4), these equations and the associated concentration factors are no longer valid. Chong and Pinter (1984) employed finite elements to investigate the effect of the ratio a/w (hole radius/width of strip) on the stress concentration factor k for the loading shown in Fig. 6-10.4. They found that in the range of a/w from 0.3 to 0.9, k varies from 3.44 to 19.50, respectively. As the ratio a/w approaches 0.99, k increases to 163. For small values of a/w (< 0.1), k becomes essentially constant and equal to approximately 3. An extensive literature survey, including experimental results, is also presented in the paper.

Problem Set 6-10

1. A very large plate has a small circular hole in it. At a long distance from the hole, $\sigma_x = 20$ kip/in.^{2}, $\sigma_y = 30$ kip/in.^{2}, $\tau_{xy} = 0$. Calculate the maximum tensile stress in the plate adjacent to the hole.

2. Consider the Airy stress function $F = f(r) \cos 2\theta$, where (r, θ) are polar coordinates and $f(r)$ is a function of r only.

(a) Derive the differential equation that defines $f(r)$.

(b) Show that $f(r) = C_1 r^2 + C_2 R^4 + C_3(1/r^2) + C_4$ is the solution of the differential equation of part (a).

(c) Consider the polar coordinate region bounded by the θ coordinate lines $r = a$, $r = b$, $a < b$. Determine the equations that define C_1, C_2, C_3, C_4, supposing that $\sigma_r = \tau_{r\theta} = 0$ for $r = a$, and $\sigma_r = \sigma \cos 2\theta$, $\tau_{r\theta} = -\sigma \sin 2\theta$ for $r = b$, where σ is a known constant.

6-11 Examples

A large number of special cases of the general solution of Eq. (6-5.5) find important practical applications in practice. Rather than discuss these cases in detail, we merely note briefly some important specializations of Eqs. (6-5.5) and the corresponding applications.

Example 6-11.1. Pure Bending of Curved Bars. The Airy stress function [see Eq. (6-6.4)]

$$F = A \log r + Br^2 \log r + Cr^2 + D \tag{E6-11.1}$$

may be used to study the problem of pure bending of curved bars (Fig. E6-11.1). The corresponding stress components are given by Eq. (6-6.5). The constants A, B, and C are determined from the boundary conditions (the beam has unit thickness)

$$\begin{aligned} \sigma_r &= 0, & r &= a, b \\ \int_a^b \sigma_\theta \, dr &= 0, & \int_a^b \sigma_\theta r \, dr &= -M \\ \tau_{r\theta} &= 0 & &\text{on all boundaries} \end{aligned} \tag{E6-11.2}$$

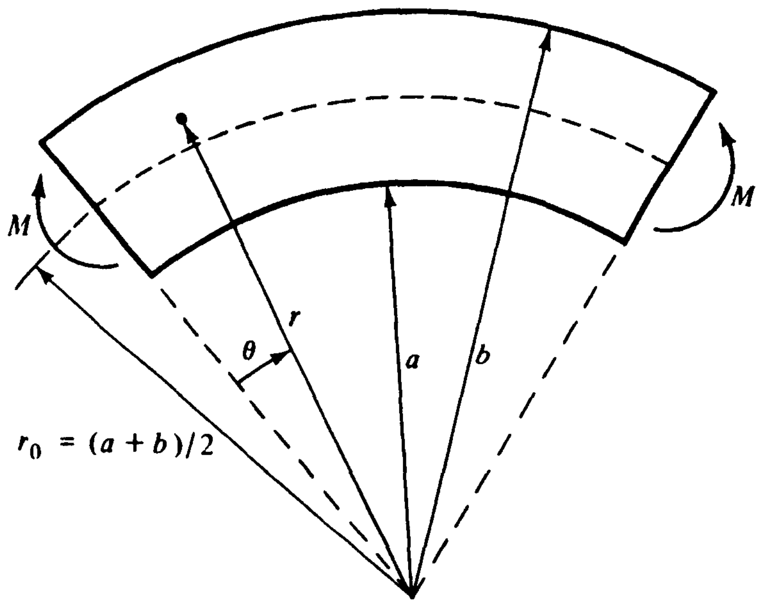


Figure E6-11.1

Hence, the stress components are then defined by Eqs. (6-6.5) with

$$\begin{aligned} A &= -\frac{4M}{N} a^2 b^2 \log\frac{b}{a} \\ B &= -\frac{2M}{N}(b^2 - a^2) \\ C &= \frac{M}{N}[b^2 - a^2 + 2(b^2 \log b - a^2 \log a)] \\ N &= (b^2 - a^2)^2 - 4a^2b^2\left(\log\frac{b}{a}\right)^2 \end{aligned} \tag{E6-11.3}$$

The strain components may be obtained for either plane strain or plane stress conditions. The displacement components may be obtained then by direct integration of the strain–displacement relations [Eqs. (6-3.5)].

Example 6-11.2. Circular Cantilever Beam. The Airy stress function

$$F(r, \theta) = f(r)\sin\theta \tag{E6-11.4}$$

may be used to study the problem of the circular cantilever beam subject to end shear (Fig. E6-11.2). With the compatibility condition $\nabla^2\nabla^2 F = 0$ and Eq. (E6-11.4), we find

$$f(r) = Ar^3 + \frac{B}{r} + Cr + Dr\log r \tag{E6-11.5}$$

Hence, by Eqs. (6-2.2), (E6-11.4), and (E6-11.5), the stress components are

$$\begin{aligned} \sigma_r &= \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta \\ \sigma_\theta &= \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta \\ \tau_{r\theta} &= -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta \end{aligned} \tag{E6-11.6}$$

With the boundary conditions

$$\sigma_r = \tau_{r\theta} = 0 \qquad \text{at} \qquad r = a, b$$

and the end shear condition

$$\int_a^b \tau_{r\theta}\, dr = P$$

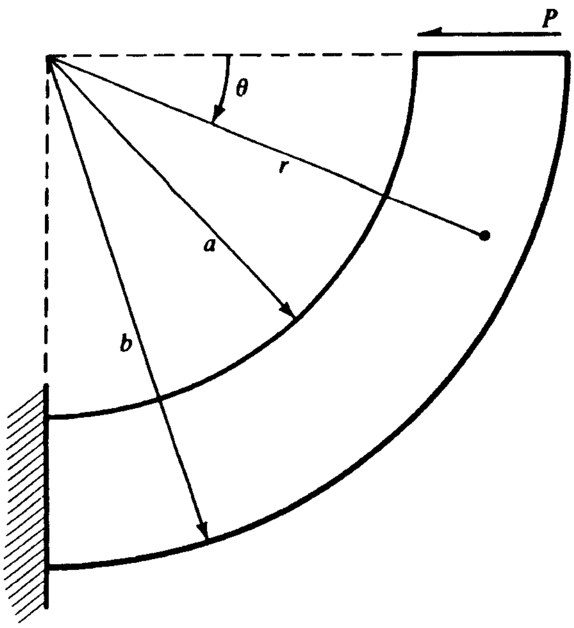


Figure E6-11.2

Eqs. (E6-11.6) yield

$$A = \frac{P}{2N}, \qquad B = -\frac{Pa^2b^2}{2N}, \qquad D = -\frac{P}{N}(a^2 + b^2)$$
$$N = a^2 - b^2 + (a^2 + b^2)\log\frac{b}{a} \tag{E6-11.7}$$

Again the strain components and the displacement components may be obtained by the equations of Section 6-4, and by integration of the strain–displacement relations [Eqs. (6-3.5)].

It may also be shown that the Airy stress function

$$F = f(r)\cos\theta \tag{E6-11.8}$$

yields a solution to the circular cantilever beam subjected to end tension T and end moment M (Fig. E6-11.3). Then, by appropriate superposition of the results obtained with Eqs. (E6-11.1),. (E6-11.4), and (E6-11.8), solutions of the problems illustrated in Figs. E6-11.4 and E6-11.5 may be obtained.

Example 6-11.3. Normal Point Load on Edge of Half-Plane. The problem of the point load P on the half-plane boundary (Fig. E6-11.6) may be analyzed by means of

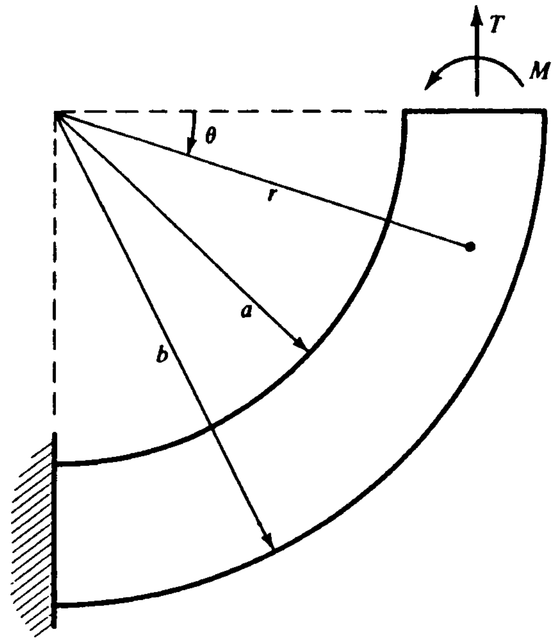


Figure E6-11.3

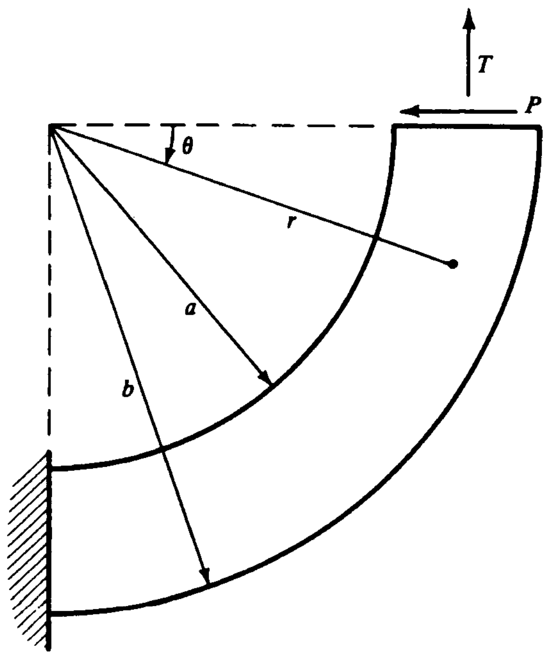


Figure E6-11.4

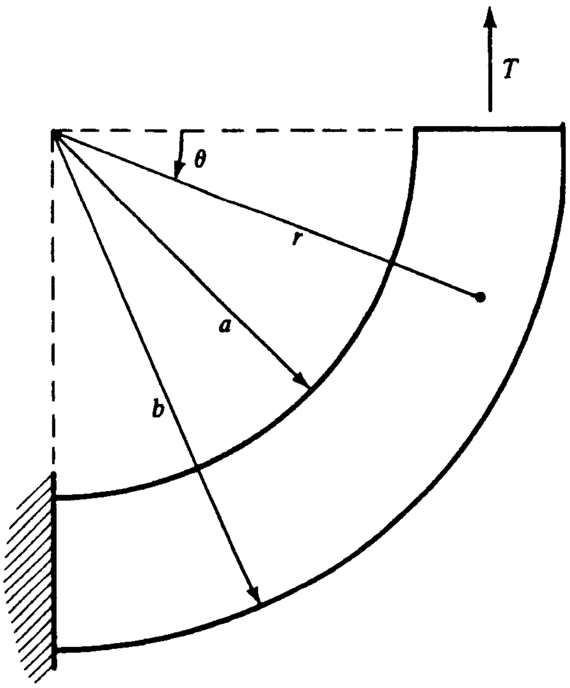


Figure E6-11.5

the stress function (under the condition that stresses vanish as $r \to \infty$):

$$F(r, \theta) = -\frac{P}{\pi} r\theta \sin\theta \qquad \text{(E6-11.9)}$$

The derivation of the stress components is left as an exercise. Note that the point $r = 0$ is a singular point (yields an infinite stress). This result may be used to obtain the stress distribution in the half-plane under the action of several point forces (see Problem Set 6-11).

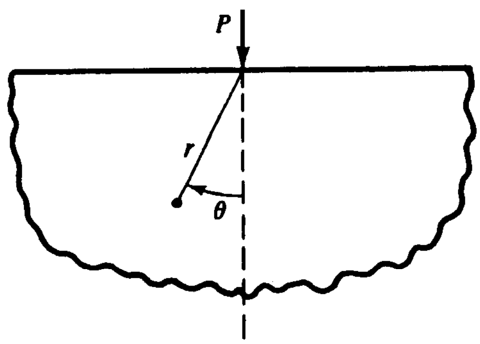


Figure E6-11.6

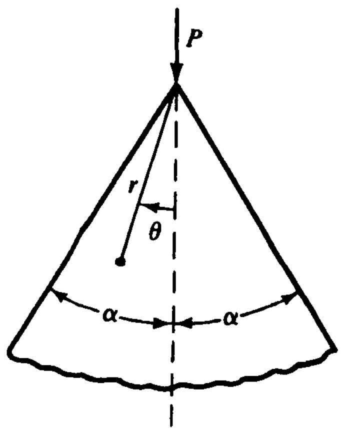


Figure E6-11.7

Example 6-11.4. Plane Wedge Under Load at Tip. The wedge problem under point load at the tip may be studied by the stress function (Fig. E6-11.7)

$$F = Ar\theta \sin\theta \qquad \text{(E6-11.10)}$$

where A is a constant determined by the boundary condition of equilibrium for a tip element. The stresses vanish as $r \to \infty$ (compare Example 6-11.3). Certain paradoxes of the wedge problem have been treated in the literature (Sternberg and Koiter, 1958; Ting, 1984a, 1984b).

Problem Set 6-11

1. Derive the strain components and the displacement components of Example 6-11.1. Assume a state of plane stress in the (r, θ) plane.
2. Repeat Problem 1 for Example 6-11.2. Assume appropriate constraints at the wall (support).
3. Repeat Problem 1 for Example 6-11.3. Discuss the behavior at $r = 0$.
4. Repeat Problem 1 for Example 6-11.4. Discuss the cases

$$\alpha = \pi/2, \qquad \pi/2 < \alpha < \pi$$

5. A circular cantilever beam is loaded in pure bending (Fig. P6-11.5). Determine the displacement of the end (point A). For $r = (a + b)/2$, $\theta = 0$, let the radial and tangential displacement components vanish, $u = v = 0$, and $\partial u/\partial\theta = 0$.
6. A thick rectangular plate is rolled into a cylindrical shape (Fig. P6-11.6). Residual stresses resulting from the rolling process are removed by annealing. After annealing, the end planes 1 and 2 are a small angle α apart. The end planes are then brought together by applying a moment M to each plane, and the faces are welded together. Then uniform internal pressure p_i and external pressure p_0 are applied to the lateral surfaces of the cylinder. Derive expressions for the radial and tangential stress components σ_r and σ_θ.

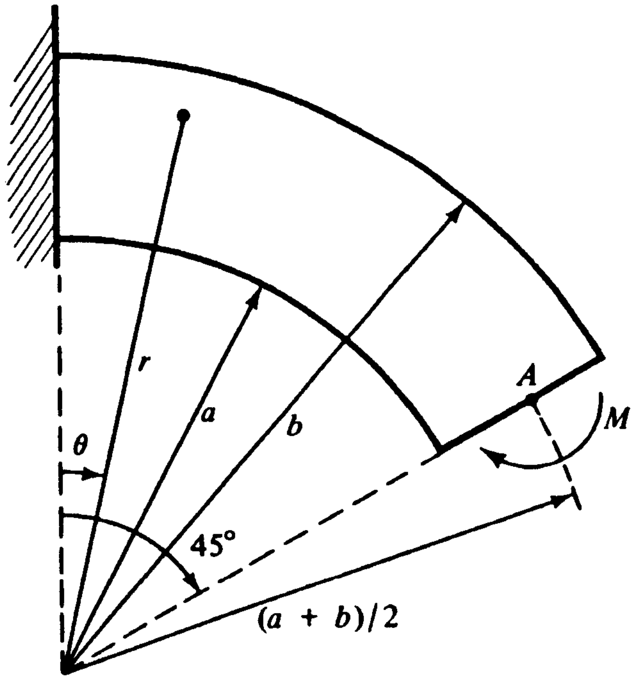


Figure P6-11.5

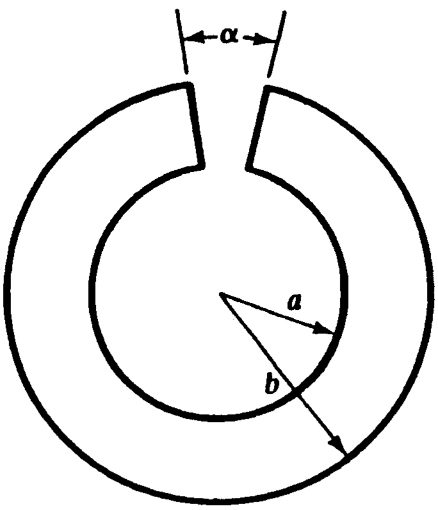


Figure P6-11.6

7. The stress function $F = Ar\theta \sin\theta$ yields the solution to the problem of a semi-infinite plate loaded by a concentrated force perpendicular to its straight-line boundary, where $0 < r$, $-\pi/2 \leq \theta \leq \pi/2$. Derive a formula for the maximum shearing stress at a point in the plate some distance from the load. Derive an equation for curves along which the maximum shearing stress is a constant, and trace several of these curves on a sketch of the plate. Derive expressions for radial and tangential components of displacement.

8. A semi-infinite plate is loaded normally to its free boundary by a concentrated force P (Fig. P6-11.8). Assume that $\sigma_\theta = \tau_{r\theta} = 0$. Hence, show that $r\sigma_r = f(\theta)$, where $f(\theta)$ is a function of θ alone. Derive the formula for $f(\theta)$. Hence, express σ_r as a known function of r, θ, and the load P. Derive expressions for radial and tangential components of displacement.

9. The stress function for a single concentrated force P acting perpendicular to the straight boundary of a semi-infinite plate is

$$F = -\frac{P}{\pi} r\theta \sin\theta$$

By the method of superposition, derive expressions for the principal stresses and the maximum shear at point A for the semi-infinite plate loaded as shown in Fig. P6-11.9, for the cases $Q = P$ and $Q = 2P$.

10. Two forces P are applied a distance $2b$ apart perpendicularly to the edge of a semi-infinite plate (Fig. P6-11.10).
    (a) Determine the principal stresses at a point D at a depth d below the surface in the line of symmetry.
    (b) The two forces P are replaced by a single force $2P$ applied in the line of symmetry. Determine the depth c below which the minimum principal stress at D is changed by less than 4 percent.

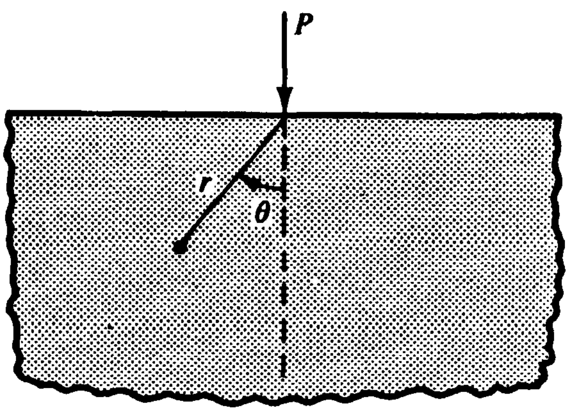


Figure P6-11.8

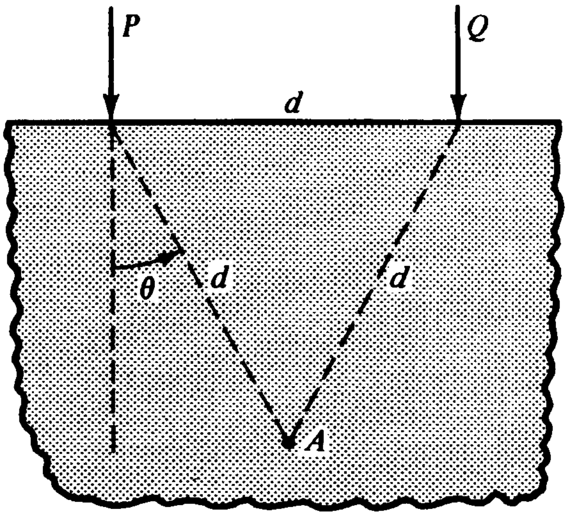


Figure P6-11.9

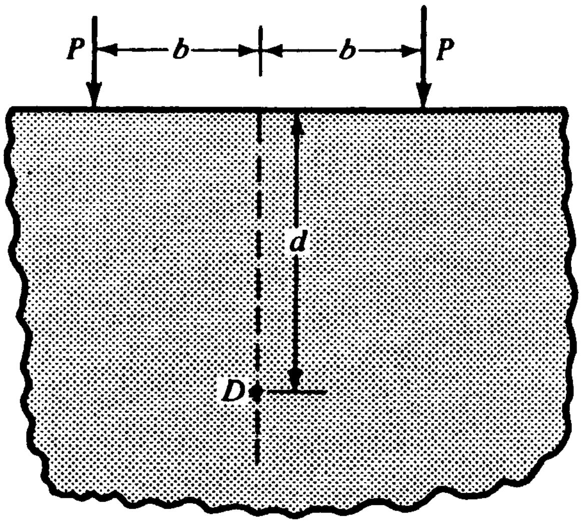


Figure P6-11.10

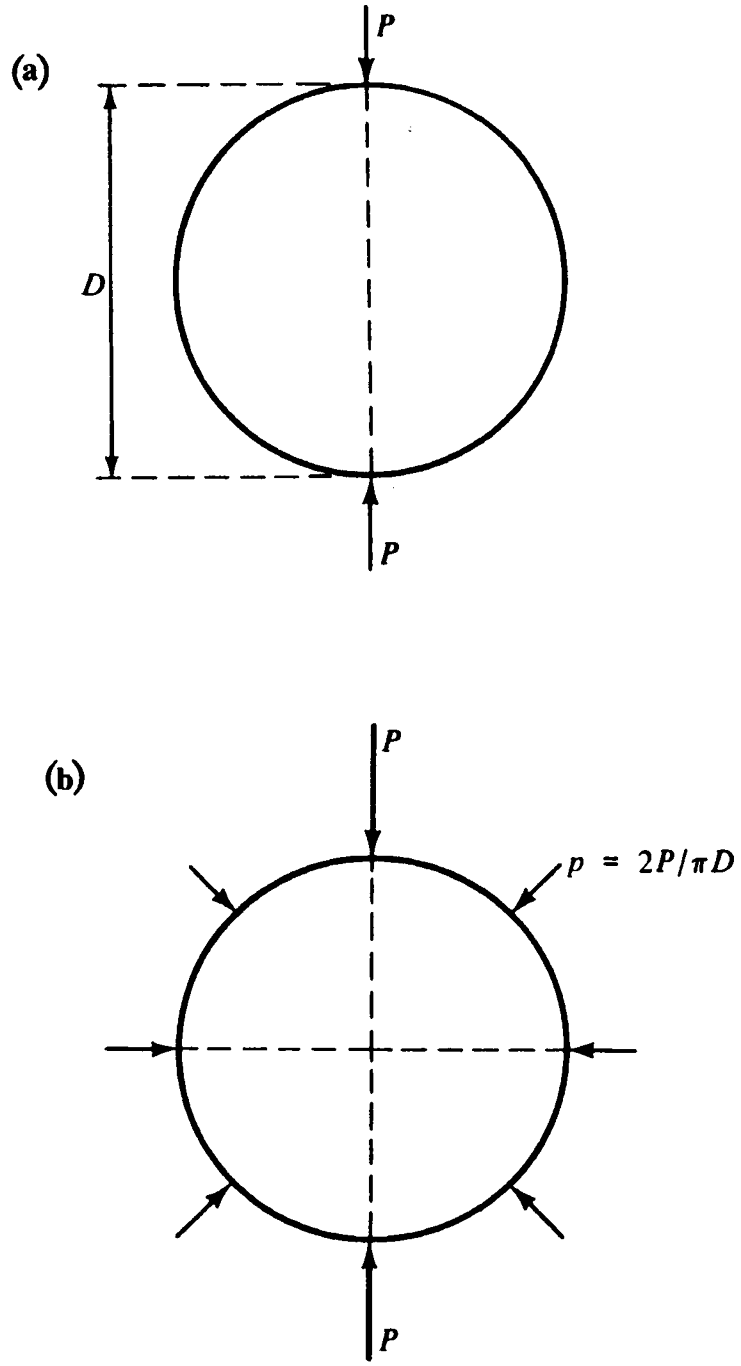


Figure P6-11.11

11. Consider a plane disk subjected to diametrically directed forces P, as shown in Fig. P6-11.11a. (See Appendix 6B for a more advanced discussion of this problem.)

(a) By considering the solution of the half-plane subjected to point load P acting normal to the straight-line boundary, show by superposition of two appropriate half-plane problems that we may obtain a solution to the disk problem for boundary stresses as shown in Fig. P6-11.11b.

(b) Then, select a state of stress that when superposed upon that of Fig. P6-11.11b, is a solution to the problem of Fig. P6-11.11a.

12. A tangential concentrated force P is applied to the upper half-plane ($y \geq 0$) at the origin (Fig. P6-11.12). Formulate the problem in terms of the Airy function. Determine the stress components. (*Hint*: See Problems 7 and 8.)

13. The semi-infinite plate is loaded uniformly along the straight-line boundary $\theta = \pi$ (Fig. P6-11.13). Show that the stress components may be derived from the stress function $F = Cr^2(\theta - \sin\theta\cos\theta)$. Evaluate the stress component for $\theta = \pi/2$; for $\theta = 0$. Discuss any discrepancies in these components. Derive expressions for radial and tangential components of displacement.

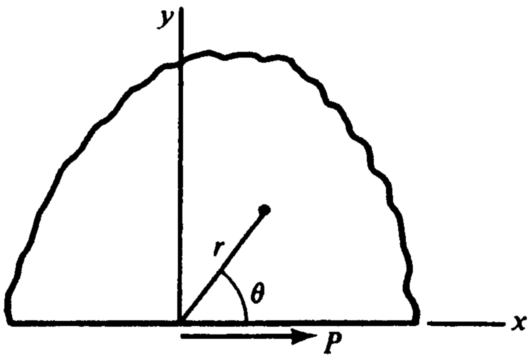


Figure P6-11.12

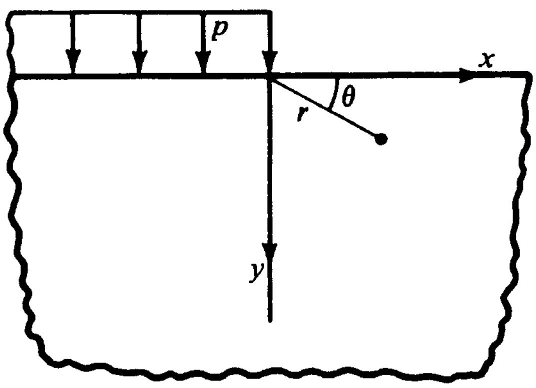


Figure P6-11.13

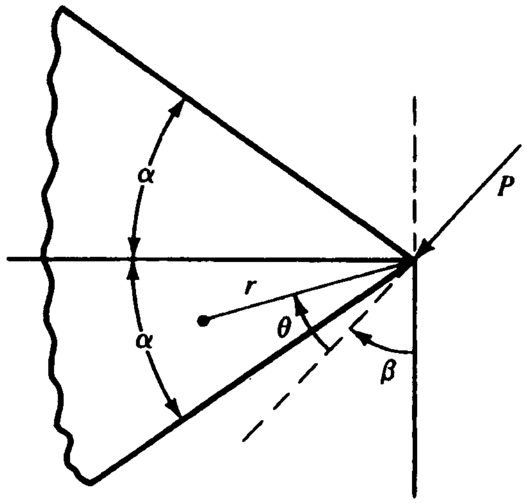


Figure P6-11.14

14. For a state of plane stress expressed in polar coordinates, assume that all stress components except σ_r are zero.

(a) In the absence of body forces and acceleration, show that $r\sigma_r = f(\theta)$, where $f(\theta)$ is an arbitrary function of θ.

(b) Derive a general formula for $f(\theta)$.

(c) Apply the results of parts (a) and (b) to the problem of a cantilever wedge loaded in its plane by a concentrated force P applied at its tip (Fig. P6-11.14); that is, express σ_r as a completely determined function of r and θ. Discuss the boundary conditions at the support.

15. A thin plate in the shape of a wedge is subjected to uniform pressure p acting along its side $\theta = -\alpha$ and a uniform pressure q acting along its side $\theta = \alpha$ $(0 < \alpha < \pi/2)$. Because $\sigma_\theta = -p, -q$ for $\theta = -\alpha, +\alpha$, $\partial^2 F/\partial r^2$ must be independent of the radial coordinate r, where (r, θ) denote polar coordinates and F is the Airy stress function. The tip of the wedge is located at $r = \theta = 0$. Hence, the Airy stress function F may at most be proportional to r^2. Accordingly, by the general solution of $\nabla^2\nabla^2 F = 0$ [see Eq. (6-5.5)], we take

$$F = (A + B\theta + C\cos 2\theta + D\sin 2\theta)r^2$$

(a) Derive the conditions that define the constants A, B, C, D.

(b) For the case $p = 0$, $\alpha = \pi/2$, show that the solution yields the case of a semi-infinite plate subjected to uniform pressure on one-half of its boundary.

16. **(a)** In the absence of body forces and temperature field, show that $F = Cr^2(2\theta - \sin 2\theta)$, $C > 0$ a constant, is an Airy stress function.

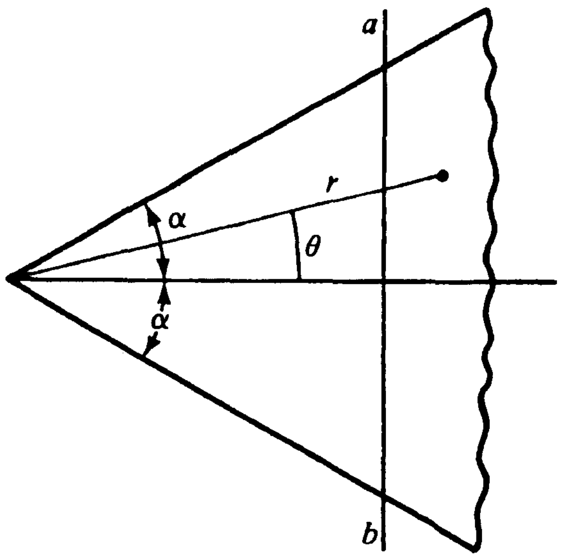


Figure P6-11.16

(b) For a plane wedge (Fig. P6-11.16), employ F of part (a) to determine possible boundary conditions for the surfaces $\theta = \pm\alpha$, where 2α is the wedge angle.

(c) In terms of the constant C and polar coordinates (r, θ), determine the explicit formulas for the stress components σ_x, τ_{xy} on the vertical section *ab*.

17. A plane wedge (tapered beam with thickness of 1 unit) is loaded at its tip by a force P (Fig. P6-11.17). In terms of polar coordinates (r, θ) the stress components are $\sigma_r = -(kP\cos\theta)/r$, $\sigma_\theta = \tau_{r\theta} = 0$, where $k = 2/(2\alpha - \sin 2\alpha)$ and α is the half-angle of the wedge. In terms of k, P, x, y, derive expressions for the stress components σ_x, σ_y, τ_{xy} relative to the rectangular Cartesian axes (x, y). Evaluate the maximum shearing stress at the point $x = 1, y = -1$. (*Hint*: Consider the equilibrium of appropriate elements or parts of the wedge.)

18. Determine the value of the constant C in the stress function

$$F = C[r^2(\alpha - \theta) + r^2 \sin\theta\cos\theta - r^2\cos^2\theta\tan\alpha]$$

required to satisfy the conditions on the upper and lower edges of the triangular plate shown in Fig. P6-11.18. Evaluate σ_x and τ_{xy} for a vertical section *mn*.

19. A stress function used in solving the problem of vertical loading of a straight boundary of a semi-infinite plane region is $F = Ar^2\theta$. Consider a point $P{:}\,(r, \theta)$ (see Fig. P6-11.19). Transform F into a function of the rectangular coordinates (x, y). Hence, derive expressions for the stress components σ_x, σ_y, τ_{xy} that act at point P. Examine the boundary conditions for $\theta = \pm\pi/2$.

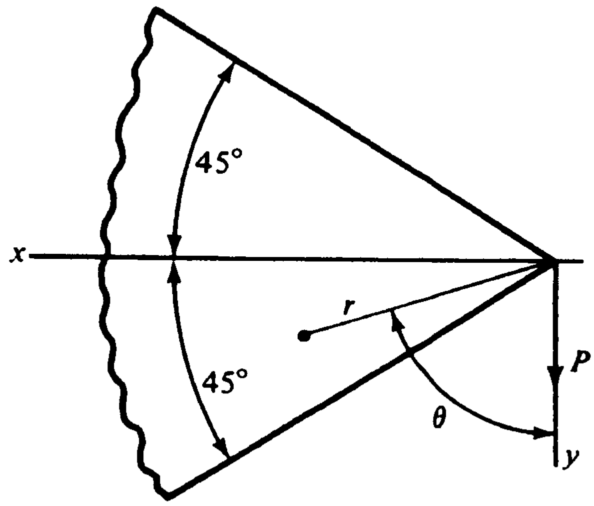


Figure P6-11.17

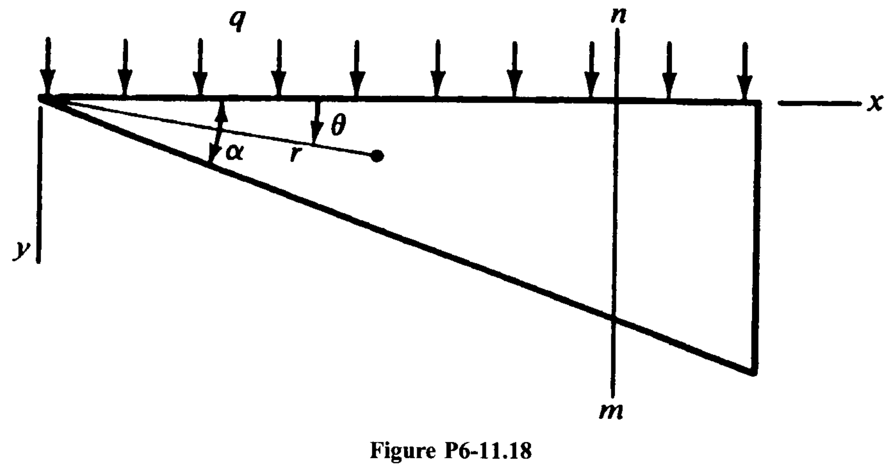


Figure P6-11.18

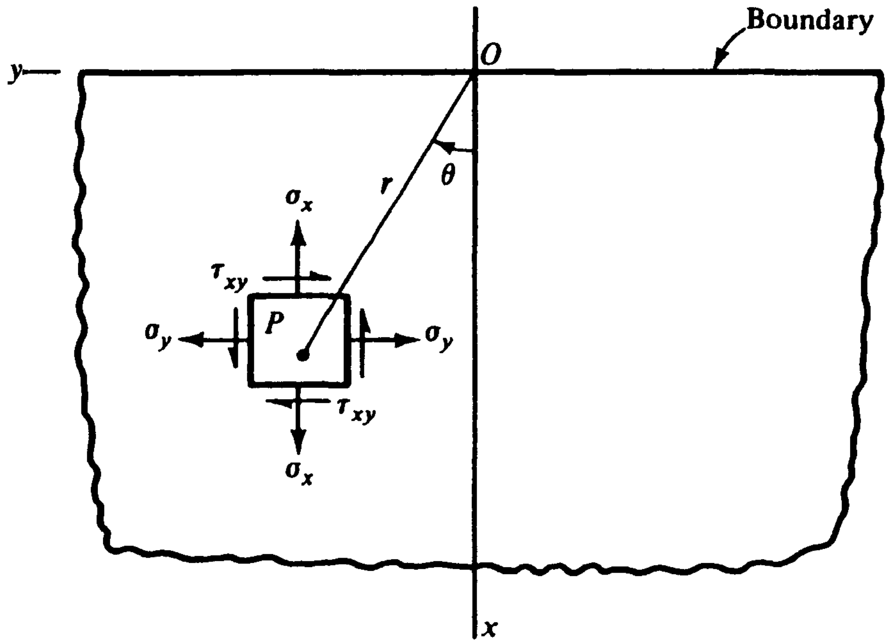


Figure P6-11.19

APPENDIX 6A STRESS-COUPLE THEORY OF STRESS CONCENTRATION RESULTING FROM CIRCULAR HOLE IN PLATE

The theory of plane elasticity with couple stresses is treated in Appendix 5A. In the present appendix we give the solution to the plane-elasticity theory with couple stresses for the circular hole in a plane region under uniform tension σ (Fig. 6-10.1). The governing equations for the function H and ψ are Eqs. (5A-5.8) and (5A-5.10). The solutions for H and ψ in polar coordinates are (see Mindlin, 1963; Weitsman, 1965; Kaloni and Ariman, 1967, Chapter 5 References)

$$
\begin{aligned}
H &= \frac{\sigma}{4} r^2(1 - \cos 2\theta) + A \log r + \left(\frac{B}{r^2} + C\right)\cos 2\theta \\
\psi &= \left[\frac{D}{r^2} + EK_2\left(\frac{r}{l}\right)\right]\sin 2\theta
\end{aligned}
\tag{6A-1}
$$

where A, B, C, D, E are constants and $K_2(r/l)$ is the modified Bessel function of the second kind and second order. Equations (6A-1) may be shown to satisfy Eqs. (5A-5.8), (5A-5.9), and (5A-5.10).

In terms of polar coordinates (r, θ), the stress components and couples are σ_r, σ_θ, $\tau_{r\theta}$, $\tau_{\theta r}$, m_{rz}, $m_{\theta z}$ (Fig. 6A-1). By equilibrium of triangular elements 1 and 2

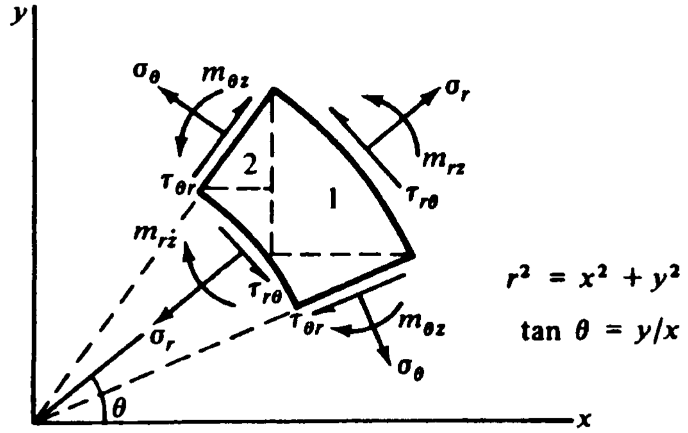


Figure 6A-1

(Figs. 6A-1 and 6A-2), we obtain

$$
\begin{aligned}
\sigma_r &= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + (\tau_{xy} + \tau_{yx}) \sin\theta\cos\theta \\
\sigma_\theta &= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - (\tau_{xy} + \tau_{yx}) \sin\theta\cos\theta \\
\tau_{r\theta} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\cos^2\theta - \tau_{yx}\sin^2\theta \\
\tau_{\theta r} &= (\sigma_y - \sigma_x)\sin\theta\cos\theta - \tau_{xy}\sin^2\theta + \tau_{yx}\cos^2\theta \\
m_{rz} &= m_{xz}\cos\theta + m_{yz}\sin\theta \\
m_{\theta z} &= -m_{xz}\sin\theta + m_{yz}\cos\theta
\end{aligned}
\tag{6A-2}
$$

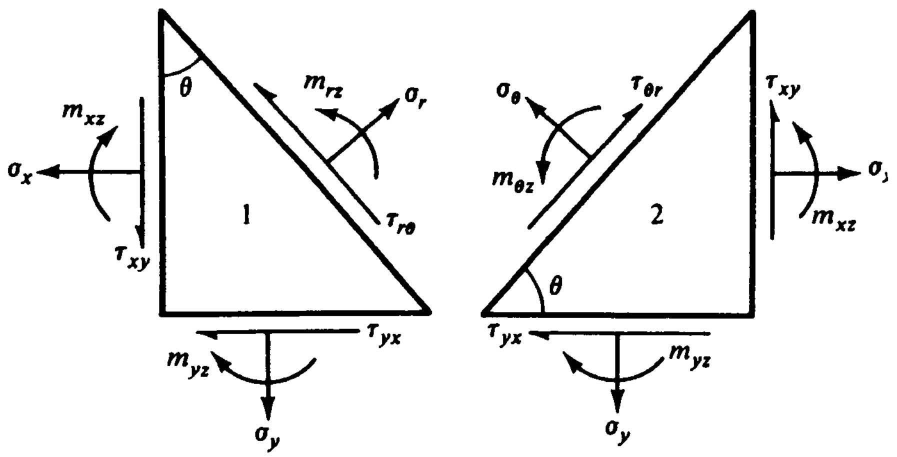


Figure 6A-2

Accordingly, by Eqs. (5A-5.7) and (6A-2) and the relations

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}
\end{aligned}
\tag{6A-3}
$$

we find in terms of H and ψ

$$
\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \\
\sigma_\theta &= \frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \\
\tau_{r\theta} &= -\frac{1}{r}\frac{\partial^2 H}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial H}{\partial \theta} - \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \\
\tau_{\theta r} &= -\frac{1}{r}\frac{\partial^2 H}{\partial r\,\partial \theta} + \frac{1}{r^2}\frac{\partial H}{\partial \theta} + \frac{\partial^2 \psi}{\partial r^2} \\
m_{rz} &= \frac{\partial \psi}{\partial r} \\
m_{\theta z} &= \frac{1}{r}\frac{\partial \psi}{\partial \theta}
\end{aligned}
\tag{6A-4}
$$

Substitution of Eqs. (6A-1) into Eq. (6A-4) yields, with the observation that

$$
\begin{aligned}
K_2\left(\frac{r}{l}\right) &= \frac{2l}{r}K_1\left(\frac{r}{l}\right) + K_0\left(\frac{r}{l}\right) \\
\frac{\partial K_0(r/l)}{\partial r} &= -\frac{1}{l}K_1\left(\frac{r}{l}\right) \\
\frac{\partial K_1(r/l)}{\partial r} &= -\frac{1}{r}K_1\left(\frac{r}{l}\right) - \frac{1}{l}K_0\left(\frac{r}{l}\right)
\end{aligned}
\tag{6A-5}
$$

where K_0 and K_1 are the modified Bessel functions of the second kind of orders zero and one (Irving and Mullineaux, 1959),

$$
\begin{aligned}
\sigma_r &= \frac{\sigma}{2}(1+\cos 2\theta) + \frac{A}{r^2} - \left(\frac{6B}{r^4} + \frac{4C}{r^2} - \frac{6D}{r^4}\right)\cos 2\theta \\
&\quad + \frac{2E}{lr}\left[\frac{3l}{r}K_0\left(\frac{r}{l}\right) + \left(1+\frac{6l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\cos 2\theta \\
\sigma_\theta &= \frac{\sigma}{2}(1-\cos 2\theta) - \frac{A}{r^2} + \left(\frac{6B}{r^4} - \frac{6D}{r^4}\right)\cos 2\theta \\
&\quad - \frac{2E}{lr}\left[\frac{3l}{r}K_0\left(\frac{r}{l}\right) + \left(1+\frac{6l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\cos 2\theta \\
\tau_{r\theta} &= -\left(\frac{\sigma}{2} + \frac{6B}{r^4} + \frac{2C}{r^2} - \frac{6D}{r^4}\right)\sin 2\theta \\
&\quad + \frac{E}{lr}\left[\frac{6l}{r}K_0\left(\frac{r}{l}\right) + \left(1+\frac{12l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\sin 2\theta \\
\tau_{\theta r} &= -\left(\frac{\sigma}{2} + \frac{6B}{r^4} + \frac{2C}{r^2} - \frac{6D}{r^4}\right)\sin 2\theta \\
&\quad + \frac{E}{l^2}\left[\left(1+\frac{6l^2}{r^2}\right)K_0\left(\frac{r}{l}\right) + \left(\frac{3l}{r} + \frac{12l^3}{r^3}\right)K_1\left(\frac{r}{l}\right)\right]\sin 2\theta \\
m_{rz} &= -\frac{2D}{r^3}\sin 2\theta - \frac{E}{l}\left[\frac{2l}{r}K_0\left(\frac{r}{l}\right) + \left(1+\frac{4l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\sin 2\theta \\
m_{\theta z} &= \left\{\frac{2D}{r^3} + \frac{2E}{r}\left[K_0\left(\frac{r}{l}\right) + \frac{2l}{r}K_1\left(\frac{r}{l}\right)\right]\right\}\cos 2\theta
\end{aligned}
\tag{6A-6}
$$

The constants A, B, C, D, E are determined by the boundary conditions

$$\sigma_r = \tau_{r\theta} = m_{rz} = 0, \qquad r = a \tag{6A-7}$$

and the condition

$$D = 8(1-\nu)l^2 C$$

where ν is Poisson's ratio, which is required for the satisfaction of Eq. (5A-5.9) in polar coordinates. Thus, we find

$$\begin{aligned}
A &= -\frac{\sigma a^2}{2}, \qquad B = -\frac{\sigma a^4(1-F)}{4(1+F)} \\
C &= \frac{\sigma a^2}{2(1+F)}, \qquad D = \frac{4(1-\nu)a^2 l^2 \sigma}{1+F} \\
E &= -\frac{palF}{(1+F)K_1(a/l)}, \\
F &= \frac{8(1-\nu)}{4 + \dfrac{a^2}{l^2} + \dfrac{2a}{l}\dfrac{K_0(a/l)}{K_1(a/l)}}
\end{aligned} \tag{6A-8}$$

The terms containing σ correspond to the stress distribution due to simple tension (Section 6-10). The terms in r diminish as r increases. Hence, the stress state at points far from the hole is due to simple tension σ.

If the couple stresses are ignored, $l = 0$. Then noting that

$$\lim_{a/l \to \infty} \frac{K_0(a/l)}{K_1(a/l)} = 1$$

we see that the stress components of Eq. (6A-6) reduce to those of Eq. (6-10.11).

With $l \neq 0$, the stress σ_θ at the hole ($r = a$) for $\theta = \pi/2,\ 3\pi/2$, is (Fig. 6-10.1)

$$\sigma_\theta = \sigma \frac{3+F}{1+F} \tag{6A-9}$$

Accordingly, if stress couples are maintained in the theory, the stress concentration factor depends both on Poisson's ratio ν and the ratio of the radius a of the hole and the material constant l [Eq. (5A-4.6)]. If couple stresses are discarded, $F = 0$, and the stress concentration factor is 3 [Eq. (6A-9)] as usual (Section 6-10). As a/l decreases, so does the stress concentration factor. With $a/l = 3$ and $\nu = 0(0.5)$, the stress concentration factor is 2.4(2.6).

Although the above theory implies that the ratio a/l influences the stress concentration factor, experiments indicate that in order to do so the material constant l must be of order of the grain size (Ellis and Smith, 1967).

Indeed, on the basis of these experiments, it may be concluded that the reduction (from 3) in stress concentration factors that is experimentally observed for small-radius notches and holes cannot be accounted for by the above simple couple-stress theory. The requirement that l must be about the order of magnitude of the grain size or smaller implies that theoretical foundations of the simple, isotropic, homogeneous continuum must be extended to examine the problem in finer detail (Ellis and Smith, 1967).

APPENDIX 6B STRESS DISTRIBUTION OF A DIAMETRICALLY COMPRESSED PLANE DISK

An experimental test of a plane disk subjected to diametrically directed forces P (Fig. P6-11.11a) is known as the split cylinder test (so called because the disk or cylinder tends to split along the line of action of forces P) or the Brazilian test. The split cylinder test is an extremely useful method for determining the tensile strengths of brittle materials that have much higher compressive strengths than tensile strengths (Chong, 1978). Typically, tensile failure will occur along the loaded diameter, splitting the cylinder (or disk) into two halves (Chong et al., 1982).

The classical theory (Timoshenko and Goodier, 1970) assumes that the line load is applied over an infinitesimally small width. If we assume a simple radial stress distributions for each force P and superimpose boundary stresses (similar to Problem 6-11.11), we find that the horizontal tensile stress (Fig. 6B-1) along the diameter is constant (Timoshenko and Goodier, 1970, and Problem 6-11.11) and equal to (for unit thickness)

$$\sigma_x = \frac{2P}{\pi D} \tag{6B-1}$$

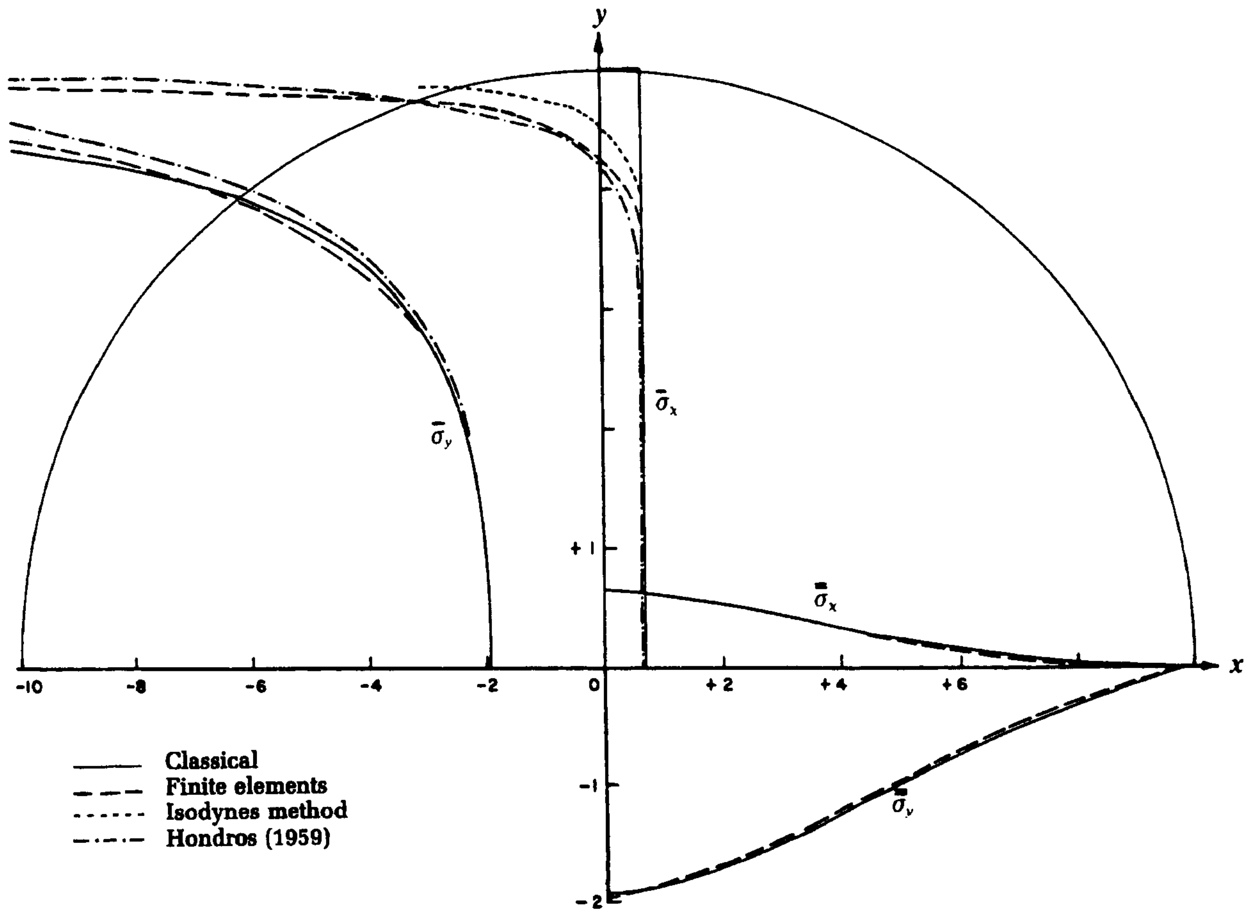
This stress distribution violates equilibrium (Fairhurst, 1964) because if half of the disk (say, left of the loaded diameter) is taken as a free body, $\sum F_x \neq 0$. To overcome these difficulties, Hondros (1959) developed a modified theory assuming negligible body forces and a finite width of loading applied radially. Numerical results based on a series solution agree closely with his experimental results monitored by strain gauges. Stresses along the loaded (vertical) diameter are given by

$$\sigma_x = \frac{2P}{\pi ab}\left[\frac{(1 - r^2/R^2)\sin 2\alpha}{(1 - 2r^2/R^2\cos 2\alpha + r^4/R^4)} - \tan^{-1}\left(\frac{1 + r^2/R^2}{1 - r^2/R^2}\tan\alpha\right)\right] \tag{6B-2}$$

$$\sigma_y = \frac{2P}{\pi ab}\left[\frac{(1 - r^2/R^2)\sin 2\alpha}{(1 - 2r^2/R^2\cos 2\alpha + r^4/R^4)} + \tan^{-1}\left(\frac{1 + r^2/R^2}{1 - r^2/R^2}\tan\alpha\right)\right]$$
$$\tau_{xy} = 0 \tag{6B-3}$$

where r = radial distance from the origin; R = radius of the disk; a = width of the applied load; and $2\alpha = a/R$. At any other point on the disk, the stresses are given in a series form. For long cylinders (plane strain case) and thin disks (plane stress case), the stress expressions given remain unchanged. However, the stress–strain relationships are different.

The finite element method can be used to model the split cylinder test (Chong et al., 1982). As a result of symmetry, only one-quarter of the disk needs to be considered. In the Chong et al. (1982) study, a total of 250 two-dimensional elements with 146 nodes were used. Each node had two degrees of freedom. The



y
x
$\bar{\sigma}_x$
$\bar{\sigma}_y$
$\bar{\bar{\sigma}}_x$
$\bar{\bar{\sigma}}_y$
-10
-8
-6
-4
-2
0
+2
+4
+6
+1
-1
-2
Classical
Finite elements
Isodynes method
Hondros (1959)

Figure 6B-1

nodal stresses were computed using consistent stress distributions. The load of $P/2$ was assumed to act at the apex node.

The stress distributions from the above theories, experiments, and the finite element method along the vertical diameter ($\bar{\sigma}_x, \bar{\sigma}_y$) and the horizontal diameter ($\bar{\bar{\sigma}}_x, \bar{\bar{\sigma}}_y$) are presented in Fig. 6B-1. These stresses have been normalized (divided) by the quantity $\sigma_0 = P/(bd)$ for comparison with other references. Four different methods are compared in the figure: (a) classical theory of Timoshenko and Goodier (1970); (b) finite element analysis (Chong et al., 1982); (c) isodynes method (Pindera et al., 1978); and (d) Hondros' theory (1959) with bearing width a equal to $\frac{1}{6}$ of the disk radius. Methods (a) and (b) are plotted for all four curves. For simplicity, methods (c) and (d) are shown only if they deviate from the classical theory.

It can be seen that the classical theory agrees well with all methods except for the tensile stress across the loaded diameter $\bar{\sigma}_x$. For $\bar{\sigma}_x$, methods (b), (c), and (d) show good agreement, indicating a very high compressive stress close to the load. This represents the reversal of stresses necessary for equilibrium and balance of internal horizontal forces. Both methods (b) and (d) indicate zero stress at 0.85 of the disk radius measuring from the center, whereas method (c) measures zero stress at 0.90 of the disk radius.

Physically, the region under the load experiences very high uniform compressive pressures in $\bar{\sigma}_x$ and $\bar{\sigma}_y$ [as indicated by Eqs. (6B-2) and (6B-3); this also can be seen from the finite element analysis]. Apparently this region wedges its way into the disk, causing an ultimate tensile failure in the brittle materials. This wedging action can be seen in the displacement contours based on finite element analysis.

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