CHAPTER 7

PRISMATIC BAR SUBJECTED TO END LOAD

In this chapter we consider the formulation of the classical problem of cylindrical elastic bars subjected to forces acting on the end planes of the bar. After developing the general theory, we examine bars of certain typical cross sections by elementary means. First, we consider the classical problem of torsion of prismatic bars after Saint-Venant. Next, we treat briefly the problem of bending of prismatic bars. The latter theory is again attributed principally to Saint-Venant.

7-1 General Problem of Three-Dimensional Elastic Bars Subjected to Transverse End Loads

Consider a cylindrical bar made of linearly elastic, homogeneous, isotropic material. Let the bar occupy the region bounded by a cylindrical lateral surface S and by two end planes distance L apart and perpendicular to the surface S (Fig. 7-1.1). The lateral surface of the bar is free of external load. The end planes of the bar are subjected to forces that satisfy equilibrium conditions of the bar as a whole. If the body forces are zero, the following sets of equations apply:

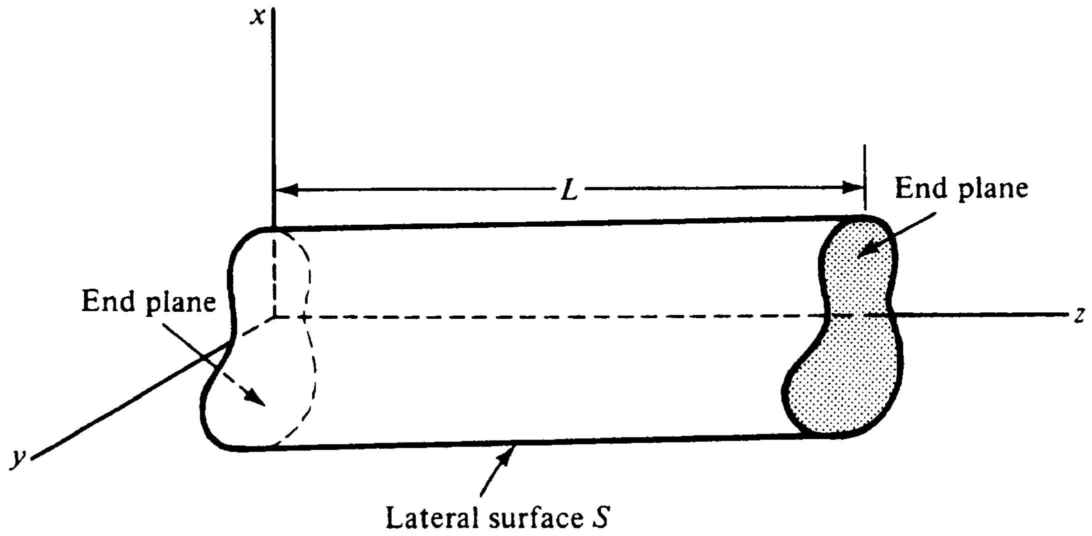


Figure 7-1.1

(a) *Equilibrium equations*:

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0\end{aligned} \tag{7-1.1}$$

(b) *Stress–strain relations*:

$$\begin{aligned}&\sigma_x = \lambda e + 2G\epsilon_x, \qquad \sigma_y = \lambda e + 2G\epsilon_y, \qquad \sigma_z = \lambda e + 2G\epsilon_z \\ &\tau_{xy} = G\gamma_{xy}, \qquad \tau_{xz} = G\gamma_{xz}, \qquad \tau_{yz} = G\gamma_{yz}\end{aligned} \tag{7-1.2}$$

or, alternatively,

$$\begin{aligned}\epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy}, \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}, \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}\end{aligned} \tag{7-1.3}$$

(c) *Boundary conditions*:
On lateral surfaces (direction cosines $l, m, n = l, m, 0$):

$$\begin{aligned}\sigma_{Px} &= l\sigma_x + m\tau_{xy} = 0 \\ \sigma_{Py} &= l\tau_{xy} + m\sigma_y = 0 \\ \sigma_{Pz} &= l\tau_{xz} + m\tau_{yz} = 0\end{aligned} \tag{7-1.4a}$$

On ends ($z = 0$, $z = L$; direction cosines $l, m, n = 0, 0, \mp 1$):

$$\tau_{xz}, \tau_{yz} \quad \text{prescribed functions} \tag{7-1.4b}$$

such that

$$\sum F_x = P_x, \qquad \sum F_y = P_y, \qquad \sum M_z = M$$

where P_x, P_y denote (x, y) components of the resultant force and M denotes the moment of the resultant couple. The problem of solving the equations formulated in the above generality poses considerable mathematical difficulties, particularly if the solution sought is to permit reasonably simple calculations. Fortunately, in a large number of practical cases, it is unnecessary to consider the problem in such general terms. Even though in practice we rarely know the true distribution of forces that act in the end planes of the bar, we often know a force system that is approximately statically equivalent to the actual force system. Accordingly, if we are considering a member with cross-sectional dimensions that are small compared to the length of the member, it may be adequate merely to ensure that the solution yields resultant forces and resultant moments that are approximately equal to actual values at the ends of the bar. For example, by Saint-Venant's principle, the stress distribution in regions sufficiently far removed from the end planes will be little affected by different distribution of forces over the end planes, provided the resultant force and moment for all distributions considered are the same (Chapter 4, Section 4-15).

Finally, the stress component σ_{ij} must satisfy the Beltrami–Mitchell compatibility equations (in the absence of body forces and for uniform temperature distribution)

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \frac{\partial^2 I_1}{\partial x_i \, \partial x_j} = 0, \qquad i, j = 1, 2, 3 \tag{7-1.5}$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_x + \sigma_y + \sigma_z \tag{7-1.6}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{7-1.7}$$

In the following discussion, we consider first the problem of twisting (torsion) of the bar by couples whose planes lie in the end planes of the bar. Then we treat the problem of bending of the bar by transverse end forces. The problems of bars subjected to axial forces at the ends and to couples whose planes are perpendicular to the end planes of the bar are left as exercises (see Review Problems R-1 and R-2 at the end of this chapter, p. 571).

7-2 Torsion of Prismatic Bars. Saint-Venant's Solution. Warping Function

In Chapter 4, Section 4-19, we treated the problem of torsion of a bar with simply connected circular cross section by the semi-inverse method. By taking displacement components in the form

$$u = -\beta yz, \qquad v = \beta xz, \qquad w = 0 \tag{7-2.1}$$

where (x, y, z) denote rectangular Cartesian coordinates and β denotes the angle of twist per unit length of the bar, we were able to satisfy the equations of elasticity exactly, provided the end shears were applied in a particular manner (Section 4-19). However, if we proceed to apply Eqs. (7-2.1) to the torsion problem of a bar with simply connected noncircular cross section, we find that in general it is not possible to satisfy the boundary conditions on the lateral surface [see Eqs. (7-14)]. Accordingly, Eqs. (7-2.1) do not represent the solution to the torsion problem of bars with noncircular cross section. Hence, we are faced with the choice of either modifying Eqs. (6-2.1) or abandoning the semi-inverse method with regard to displacement components. For example, one may attempt to add more generality to Eqs. (7-2.1) (after Saint-Venant) or one may attempt to reformulate the problem in terms of stress components (after Prandtl). Initially, in this section, we modify Eqs. (7-2.1). In Section 7-3 we return to the formulation of the problem in terms of stress components.

The concept of allowing a section distance z from the end $z = 0$ to rotate as a rigid body about the axis of twist (the z axis, Fig. 7-2.1) is analytically attractive. Accordingly, we retain the same form for (u, v) [see Eq. (7-2.1) and Section 4-19]; however, we relax the condition $w = 0$.

Because the end forces tend to twist the bar about the z axis, physically it seems reasonable that extension of the bar along its axis is of secondary importance. Hence, the dependency of w, the displacement component in the z direction, upon z appears to be of secondary importance. Physically, the dependency of w upon coordinates (x, y) is difficult to guess. Accordingly, we do not attempt to specify an explicit relation between w and (x, y): rather, we arbitrarily take (after Saint-Venant) w in the form $w = \beta\psi(x, y)$, where $\psi(x, y)$ is an arbitrary function of (x, y). Because $\psi(x, y)$ is a measure of how much a point in the plane $z =$ constant displaces in the z direction, it is called the *warping function*. Thus, for the small-displacement torsion problem of a bar with noncircular cross section, we take the displacement vector (u, v, w) in the form

$$u = -\beta zy, \qquad v = \beta zx, \qquad w = \beta\psi(x, y) \tag{7-2.2}$$

We now proceed to determine whether the equations of elasticity may be satisfied by this assumption. In other words, we seek to determine the function $\psi(x, y)$ such that the equations of elasticity are satisfied.

For small-displacement theory, Eqs. (2-15.14) and (7-2.2) yield

$$\begin{aligned} &\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \\ &\gamma_{xz} = \beta\left(\frac{\partial\psi}{\partial x} - y\right), \qquad \gamma_{yz} = \beta\left(\frac{\partial\psi}{\partial y} + x\right) \end{aligned} \tag{7-2.3}$$

Substitution of Eqs. (7-2.3) into Eqs. (7-1.2) yields the stress components

$$\begin{aligned} &\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \\ &\tau_{xz} = G\beta\left(\frac{\partial\psi}{\partial x} - y\right), \qquad \tau_{yz} = G\beta\left(\frac{\partial\psi}{\partial y} + x\right) \end{aligned} \tag{7-2.4}$$

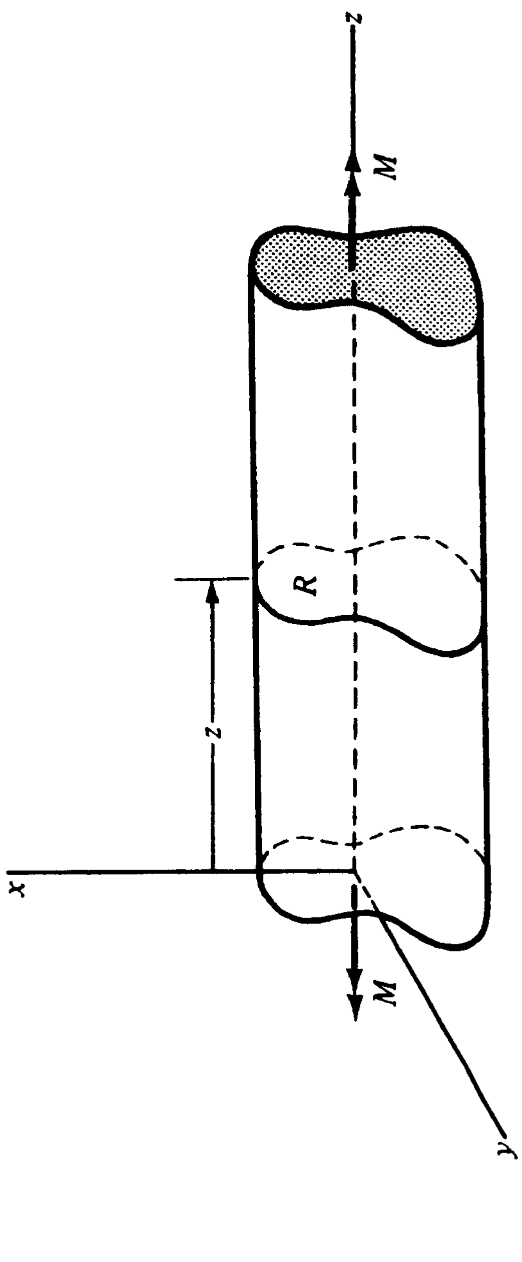


Figure 7-2.1

Now substitution of Eqs. (7-2.4) into Eqs. (7-1.1) yields for equilibrium

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0 \tag{7-2.5}$$

where now

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Accordingly, the assumption of displacement components in the form of Eqs. (7-2.2) yields the requirement that $\nabla^2 \psi = 0$, that is, that ψ be harmonic over the region R of the cross section of the bar (Fig. 7-2.1). Because we have assumed displacement components (u, v, w), compatibility conditions are automatically satisfied (Chapter 2, Section 2-16). Consequently, we have satisfied the equations of elasticity, provided that we can find a harmonic function (warping function) ψ that by Eqs. (7-2.4) yields stress components that satisfy the boundary conditions [Eqs. (7-14)].

Substituting Eqs. (7-2.4) into the boundary conditions for the lateral surface, we see that the first two of Eqs. (7-1.4a) are satisfied identically. The third equation yields

$$\left(\frac{\partial \psi}{\partial x} - y\right) l + \left(\frac{\partial \psi}{\partial y} + x\right) m = 0 \tag{7-2.6}$$

where (l, m) denote the components of the unit normal vector to the lateral surface S bounding the simply connected region R (Fig. 7-2.2). By Fig. 7-2.2, we find

$$\begin{aligned} l &= \cos \phi = \frac{dy}{ds} \\ m &= \sin \phi = -\frac{dx}{ds} \end{aligned} \tag{7-2.7}$$

Substitution of Eq. (7-2.7) into Eq. (7-2.6) yields

$$\frac{\partial \psi}{\partial x}\frac{dy}{ds} - \frac{\partial \psi}{\partial y}\frac{dx}{ds} = x\frac{dx}{ds} + y\frac{dy}{ds} = \frac{1}{2}\frac{d}{ds}(x^2 + y^2) \tag{7-2.8}$$

Furthermore, by Fig. 7-2.2, we have

$$\frac{dx}{ds} = \frac{dx}{dn}, \quad \frac{dx}{ds} = -\frac{dy}{dn} \tag{7-2.9}$$

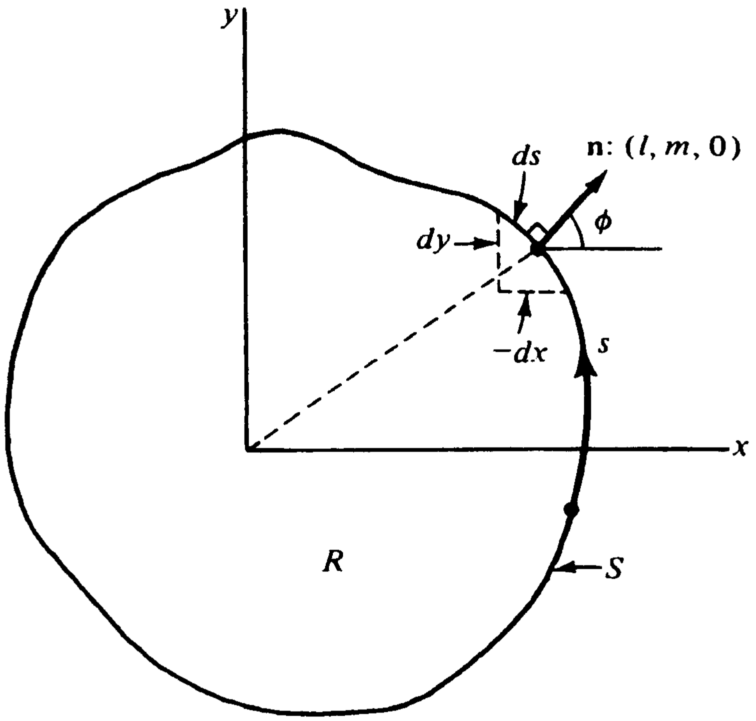


Figure 7-2.2

Consequently, Eqs. (7-2.8) and (7-2.9) yield

$$\frac{d\psi}{dn} = \frac{\partial\psi}{\partial x}\frac{dx}{dn} + \frac{\partial\psi}{\partial y}\frac{dy}{dn} = \frac{1}{2}\frac{d}{ds}(x^2 + y^2) \tag{7-2.10}$$

For a circular cross section of radius $a, x^2 + y^2 = a^2 =$ constant. Then Eq. (7-2.10) yields $d\psi/dn = 0$ on S, or $\psi =$ constant on S. This result agrees with that obtained in Section 4-19.

In general, we note that if the cross section is noncircular Eqs. (7-2.6), (7-2.7), and (7-2.9) yield the result

$$\frac{d\psi}{dn} = yl - xm = f(s) \tag{7-2.11}$$

where $f(s)$ denotes a function of the parameter s on the bounding curve S (Fig. 7-2.2).

Finally, it may be shown (see the problem at the end of this section) that

$$\sum F_x = \int_A \tau_{xz}\, dA = 0$$
$$\sum F_y = \int_A \tau_{yz}\, dA = 0 \tag{7-2.12}$$
$$\sum M_z = \int_A (x\tau_{yz} - y\tau_{xz})\, dA = M$$

Accordingly, we have obtained a solution of the torsion problem of a bar with simply connected cross section, provided $\psi(x, y)$ satisfies the equations

$$\nabla^2 \psi = 0 \qquad \text{in } R$$
$$\frac{d\psi}{dn} = yl - xm = f(s) \qquad \text{on } S \tag{7-2.13}$$

Equations (7-2.13) define a well-known, extensively studied problem of potential theory (Kellogg, 1969): The Neumann boundary-value problem.[1] In other words, the torsion problem expressed in terms of the warping function $\psi(x, y)$ may be stated as follows:

Determine a function $\psi(x, y)$ that is harmonic ($\nabla^2\psi = 0$) in R, such that it is regular in R and continuous in $R + S$, and such that its normal derivative takes on prescribed values $f(s)$ on S.

Alternatively, Eqs. (7-2.13) may be reformulated by utilizing the complex conjugate of $\psi(x, y)$, that is, by utilizing the function $\chi(x, y)$ related to $\psi(x, y)$ by the Cauchy–Riemann equation (Churchill et al., 1989)[2]:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \chi}{\partial y}, \qquad \frac{\partial \psi}{\partial y} = -\frac{\partial \chi}{\partial x} \tag{7-2.14}$$

Differentiating the first of Eqs. (7-2.14) by y, the second by x, and subtracting, we obtain $\nabla^2\chi = 0$. Substitution of Eqs. (7-2.14) and (7-2.9) into the second of Eqs. (7-2.13) yields

$$\frac{d\chi}{ds} = yl - xm = \frac{1}{2}\frac{d}{ds}(x^2 + y^2) \qquad \text{or} \qquad \chi = \frac{1}{2}(x^2 + y^2) + \text{const}$$

[1] A solution ψ to the Neumann problem exists, provided that the integral of the normal derivative of the function ψ, calculated over the entire boundary S, vanishes. Then the solution ψ is determined to within an arbitrary constant. For the torsion problem [Eqs. (7-2.13)], the solution ψ exists (see Problem 7-2.1).

[2] See also Eqs. (5-5.3) in Chapter 5.

Accordingly, in terms of the complex conjugate χ of ψ, Eqs. (7-2.13) may be written

$$\begin{aligned} \nabla^2\chi &= 0 \qquad \text{in } R \\ \chi &= \frac{1}{2}(x^2+y^2) = g(s) \qquad \text{on } S \end{aligned} \tag{7-2.15}$$

where the constant in the second equation has been set equal to zero, as it does not affect the state or stress or displacement [see Eqs. (7-2.3), (7-2.4), and (7-2.14)].

In terms of χ, the strain and stress components are, by Eqs. (7-2.3), (7-2.4), and (7-2.14),

$$\gamma_{xz} = \beta\left(\frac{\partial\chi}{\partial y} - y\right), \qquad \gamma_{yz} = -\beta\left(\frac{\partial\chi}{\partial x} - x\right) \tag{7-2.16}$$

and

$$\tau_{xz} = G\beta\left(\frac{\partial\chi}{\partial y} - y\right), \qquad \tau_{yz} = -G\beta\left(\frac{\partial\chi}{\partial x} - x\right) \tag{7-2.17}$$

The boundary-value problem defined by Eq. (7-2.15), that of seeking a harmonic function χ in region R, whose values are prescribed on the boundary S of R, is known as the Dirichlet problem. The Dirichlet problem has been studied extensively (Kellogg, 1969; Courant and Hilbert, 1989).

Problem Set 7-2

1. Verify the first two of Eqs. (7-2.12). Verify that a solution ψ to the Neumann problem exists for the torsion of a bar [see Eqs. (7-2.13)].

7-3 Prandtl Torsion Function

In the preceding section we formulated the torsion problem of the bar with simply connected cross section in terms of two associated boundary-value problems [see Eqs. (7-2.13) and (7-2.15)]. In this section we consider an alternative approach originally formulated by Prandtl (1903).[3] Prandtl employed the semi-inverse procedure as follows.

Because in the classical torsion problem the lateral surface and the end planes of the bar are free from normal tractions, one might initially guess that the normal tractions are zero throughout the bar. Furthermore, because the end faces are subjected to shear stress components that produce a couple **M**, one might initially

[3] As we will see, the results obtained by Prandtl are related simply to those obtained by Saint-Venant.

assume as a first guess that the shear component not associated with the couple **M** also vanishes. Then one has (with respect to x, y, z axes designated in Fig. 7-2.1)

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \tag{7-3.1}$$

Next, because the left and right end planes are loaded identically, it appears reasonable that the remaining two components of stress (τ_{xz}, τ_{yz}) are approximately independent of the axial coordinate z. Accordingly, assuming that τ_{xz}, τ_{yz} are functions of (x, y) only and substituting Eqs. (7-3.1) into Eqs. (7-1.1), we find

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{7-3.2}$$

Equation (7-3.2) represents the necessary and sufficient condition that there exist a function $\phi(x, y)$ such that (see Chapter 1, Section 1-19)

$$\tau_{xz} = \frac{\partial \phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} \tag{7-3.3}$$

where here the function ϕ is called the *Prandtl torsion function*.

Equation (7-3.3) automatically satisfies the equation of equilibrium [Eq. (7-3.2)]. Substitution of Eqs. (7-3.1) and (7-3.3) into Eqs. (7-1.5) yields

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = c = \text{constant} \tag{7-3.4}$$

Accordingly, compatibility is satisfied provided $\nabla^2 \phi = c$. The constant c may be shown to have a physical significance in that it is related to the angle of twist. Before verifying this statement, we consider the boundary conditions on the lateral surface and on the end planes [Eqs. (7-1.4)]. The first two of Eqs. (7-1.4a) are satisfied automatically; the last of Eqs. (7-1.4a), with Eqs. (7-2.7) and (7-3.3), yields (see Fig. 7-2.2)

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} = 0 \qquad \text{on } S$$

or

$$\phi = K = \text{constant} \qquad \text{on } S \tag{7-3.5}$$

where K denotes an arbitrary constant. For the simply connected cross section we may set $K = 0$ (see Section 7-6).

Finally, substitution of Eqs. (7-2.7) and (7-3.3) into Eqs. (7-1.4b) yields the following integrations over the end planes:

$$\begin{aligned}
\sum F_x &= \iint \sigma_{Px}\, dx\, dy = \iint \tau_{xz}\, dx\, dy \\
&= \int dx \int \frac{\partial \phi}{\partial y} dy = \int \phi \Big|_{y_1}^{y_2} dx \\
\sum Fy &= \iint \sigma_{Py}\, dx\, dy = \iint \tau_{yz}\, dx\, dy \\
&= -\int dy \int \frac{\partial \phi}{\partial x} dx = -\int \phi \Big|_{x_1}^{x_2} dy \\
\sum M_z = M &= \iint (x\tau_{yz} - y\tau_{xz})\, dx\, dy \\
&= -\iint \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right) dx\, dy \\
&= -\int x\phi \Big|_{x_1}^{x_2} dy - \int y\phi \Big|_{y_1}^{y_2} dx + 2\iint \phi\, dx\, dy
\end{aligned}$$

Because $\phi = \text{constant}$ on the lateral surface [we take $K = 0$ for the simply connected region; see Eq. (7-3.5)] and x_1, x_2, y_1, y_2 denote points on the lateral surface, it follows that

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum M_z = M = 2\iint \phi\, dx\, dy \tag{7-3.6}$$

By the above discussion, we see that the torsion problem for a simply connected cross section R is solved precisely, provided we obtain a function ϕ such that

$$\begin{aligned}
\nabla^2 \phi = c = \text{const} \qquad &\text{in } R \\
\phi = 0 \qquad &\text{on } S
\end{aligned} \tag{7-3.7}$$

and provided the shears τ_{xz}, τ_{yz} are distributed over the end planes in accordance with Eq. (7-3.3). The twisting moment M is then defined by Eq. (7-3.6). The constant c may be related to the angle of twist per unit length of the bar, as we now proceed to show.

Displacement Components. Substitution of Eqs. (7-3.1) and (7-3.3) into the stress–strain relations [Eqs. (7-1.3)] yields with Eqs. (2-15.14)

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0, \qquad & \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \\
\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{G}\tau_{xz}, \qquad & \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{G}\tau_{yz}
\end{aligned} \tag{7-3.8}$$

Integration of Eqs. (7-3.8) yields

$$u = -Az(y - b), \qquad v = Az(x - a) \tag{7-3.9}$$

where A is a constant of integration and where $x = a$, $y = b$ defines the *center of twist*, that is, the z axis about which each cross section rotates as a rigid body (see Section 4-19; there, $a = y = 0$ denotes the axis of twist).

Substitution of Eqs. (7-3.9) into the last two of Eqs. (7-3.8) yields

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{1}{G}\tau_{xz} + A(y - b)\\ \frac{\partial w}{\partial y} &= \frac{1}{G}\tau_{yz} - A(x - a)\end{aligned} \tag{7-3.10}$$

Integration of Eqs. (7-3.10) yields

$$w = w_0 - A(xb - ya) \tag{7-3.11}$$

where $w_0 = w_0(x, y)$ represents the warping of the cross section. The terms involving the constants (a, b) in Eqs. (7-3.9) and (7-3.11) represent a rigid-body displacement relative to the center of twist.

To determine the angle of twist per unit length of the bar, we recall that the rotation ω_z of a volume element relative to the z axis is [see Eqs. (2-13.2)]

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{7-3.12}$$

Substitution of Eqs. (7-3.9) into Eqs. (7-3.12) yields $\omega_z = Az$. Hence, the angle of twist β per unit length of the bar is

$$\beta = \frac{\partial \omega_z}{\partial z} = A \tag{7-3.13}$$

Therefore, the constant of integration A in Eqs. (7-3.9) is identical to the angle of twist per unit length of the bar. Furthermore, by the last two of Eqs. (7-3.8), we note that by differentiating γ_{xz} by y and γ_{yz} by x and subtracting, we obtain

$$2\beta = 2\frac{\partial \omega_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{G}\left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right) \tag{7-3.14}$$

Hence, substitution of Eq. (7-3.3) into Eq. (7-3.14) yields [with Eq. (7-3.7)]

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = c = -2G\beta \tag{7-3.15}$$

Accordingly, in terms of the Prandtl stress function ϕ, the torsion problem of a bar with simply connected cross section R bounded by S is defined by

$$\begin{aligned} \nabla^2\phi &= -2G\beta \qquad \text{in } R \\ \phi &= 0 \qquad \text{on } S \end{aligned} \tag{7-3.16}$$

For the case where $a = b = 0$, the warping displacement $w_0(x, y)$ is related to the warping function $\psi(x, y)$ by the equation [see Eqs. (7-2.2) and (7-3.11)]

$$w_0 = \beta\psi(x, y) \tag{7-3.17}$$

Furthermore, the Prandtl stress function $\phi(x, y)$ is related to the warping function $\psi(x, y)$ by the equation [see Eqs. (7-2.4) and (7-3.3)]

$$\frac{\partial\phi}{\partial y} = G\beta\left(\frac{\partial\psi}{\partial x} - y\right), \qquad \frac{\partial\phi}{\partial x} = -G\beta\left(\frac{\partial\psi}{\partial y} + x\right) \tag{7-3.18}$$

and to the complex conjugate χ of ψ by the relations [see Eqs. (7-2.14), (7-3.3), and (7-3.18)]

$$\frac{\partial\phi}{\partial y} = G\beta\left(\frac{\partial\chi}{\partial y} - y\right), \qquad \frac{\partial\phi}{\partial x} = G\beta\left(\frac{\partial\chi}{\partial x} - x\right) \tag{7-3.19}$$

Integration of these relations yields

$$\phi = G\beta[\chi - \tfrac{1}{2}(x^2 + y^2) + b] \tag{7-3.20}$$

where b denotes a constant. Thus, the Prandtl stress function ϕ may be simply related to the Saint-Venant warping function ψ [Eqs. (7-3.18)] or to the conjugate harmonic function χ of ψ [Eq. (7-3.20)].

Problem Set 7-3

1. Show that cylinders with circular cross sections are the only bodies whose lateral surface can be free from external load when the stress components are characterized by

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \qquad \tau_{xz} = -G\beta y, \qquad \tau_{yz} = G\beta x$$

7-4 A Method of Solution of the Torsion Problem: Elliptic Cross Section

A direct approach to the solution of the torsion problem is difficult in most practical cases. However, in terms of Prandtl's stress function ϕ, the following indirect approach is sometimes useful, although it is not generally applicable.

Because $\phi = 0$ on the lateral boundary [Eq. (7-3.16)], we may seek stress functions ϕ_i such that $\phi_i = 0$ on the lateral boundary of the shaft, leaving sufficient arbitrariness in ϕ so that the equation $\nabla^2\phi = -2G\beta$ may be satisfied over the region R occupied by the cross section. For a certain class of cross sections with boundaries simply expressible in the form $f(x, y) = 0$, this procedure is sometimes fruitful.

Example 7-4.1. Bar with Elliptical Cross Section. The equation of the bounding curve C of a bar with elliptical cross section is (Fig. 7-4.1)

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \tag{7-4.1}$$

Hence, if we assume a stress function ϕ in the form

$$\phi = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{7-4.2}$$

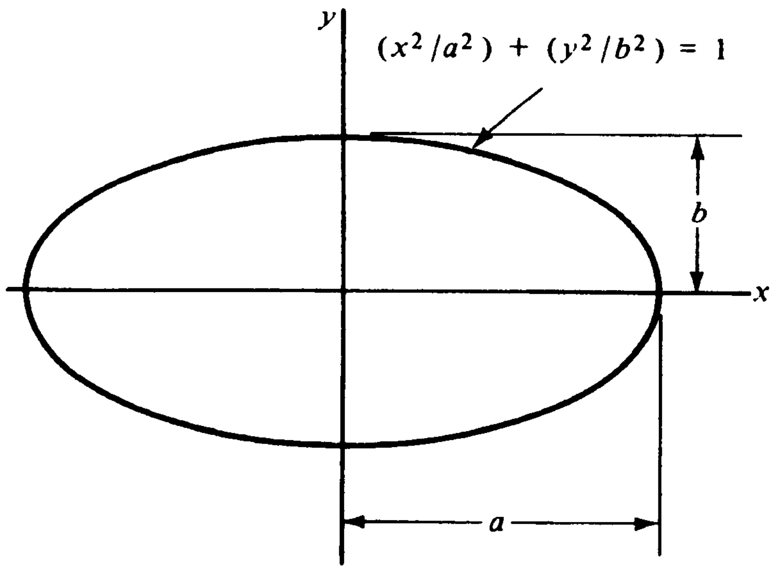


Figure 7-4.1

where A is a constant, the boundary condition $\phi = 0$ on C is automatically satisfied. To yield a solution to the torsion problem, the function ϕ must be chosen so that both of Eqs. (7-3.16) are satisfied. By Eq. (7-4.2) we find that

$$\nabla^2\phi = 2A\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Hence, in order that ϕ satisfy Eq. (7-3.16) we must have

$$A = -\frac{a^2 b^2 G\beta}{a^2 + b^2} \tag{7-4.3}$$

Accordingly, if A is given by Eq. (7-4.3), Eq. (7-4.2) yields the solution of the torsion of a bar with elliptic cross section. With ϕ so determined, the theory of Section 7-3 yields the stress components (τ_{xz}, τ_{yz}) and the moment M in terms of the dimensions a, b of the cross section, the shear modulus G, and the angle of twist β per unit length of the bar.

Moment–Angle of Twist Relation. The moment–stress function relation [Eq. (7-3.6)], with Eqs. (7-4.2) and (7-4.3), now yields

$$M = -\frac{2G\beta a^2 b^2}{a^2 + b^2}\left[\frac{1}{a^2}\iint x^2\,dx\,dy + \frac{1}{b^2}\iint y^2\,dx\,dy - \iint dx\,dy\right] \tag{7-4.4}$$

Now, for the ellipse,

$$\begin{aligned} \iint x^2\,dx\,dy &= I_y = \frac{\pi a^3 b}{4} \\ \iint y^2\,dx\,dy &= I_x = \frac{\pi a b^3}{4} \\ \iint dx\,dy &= \pi ab \end{aligned} \tag{7-4.5}$$

where (I_x, I_y) denote the moment of inertia of the cross-sectional area with respect to the (x, y) axes, respectively. Consequently, Eqs. (7-4.4) and (7-4.5) yield

$$M = \frac{\pi G\beta a^3 b^3}{a^2 + b^2} = C\beta \tag{7-4.6}$$

where

$$C = \frac{\pi a^3 b^3 G}{a^2 + b^2} \tag{7-4.7}$$

is called the *torsional rigidity* of the bar. Equation (7-4.6) relates the twisting moment M to the angle of twist β, the constant of proportionality being C, the torsional rigidity.

Also, by Eqs. (7-4.3) and (7-4.6), we find

$$A = -\frac{M}{\pi ab} \tag{7-4.8}$$

Therefore, we may write ϕ in the form

$$\phi = -\frac{M}{\pi ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{7-4.9}$$

Stress Components. By Eqs. (7-3.3) and (7-4.9), we obtain

$$\begin{aligned} \tau_{xz} &= \frac{\partial \phi}{\partial y} = -\frac{2M}{\pi ab^3}y \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} = \frac{2M}{\pi a^3 b}x \end{aligned} \tag{7-4.10}$$

Hence, (τ_{xz}, τ_{yz}) vary linearly over the cross section with respect to (y, x), respectively. To determine the direction of the shearing stress vector $\boldsymbol{\tau} = \mathbf{i}\tau_{xz} + \mathbf{j}\tau_{yz}$ on the boundary of the shaft, we note that the tangent of the angle between the vector $\boldsymbol{\tau}$ and the positive x axis is given by [Eq. (7-4.10)]

$$\frac{\tau_{yz}}{\tau_{xz}} = -\frac{b^2 x}{a^2 y} \tag{7-4.11}$$

However, by the equation of the bounding curve C of the cross section [Eq. (7-4.1)], we see that the angle formed by the tangent to C and the positive x axis is

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \tag{7-4.12}$$

Equations (7-4.11) and (7-4.12) show that the shearing-stress vector $\boldsymbol{\tau}$ is tangent to the boundary C of the cross section. For $x = a$, $y = 0$, $\boldsymbol{\tau} = \mathbf{j}\tau_{yz}$; hence, $\boldsymbol{\tau}$ is directed perpendicular to the x axis. For $x = 0$, $y = b$, $\boldsymbol{\tau} = \mathbf{i}\tau_{xz}$; then $\boldsymbol{\tau}$ is directed perpendicular to the y axis (see Fig. 7-4.2). Also, the magnitude of $\boldsymbol{\tau}$ is

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \frac{2M}{\pi ab}\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}} \tag{7-4.13}$$

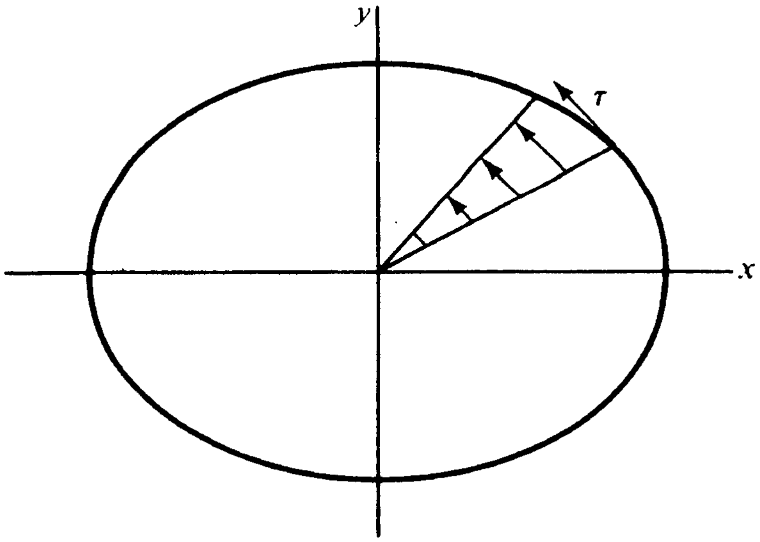


Figure 7-4.2

Determining the maximum value of τ from Eq. (7-4.13), we find

$$\tau_{\max} = \frac{2M}{\pi ab^2}, \qquad y = b, \qquad x = 0 \tag{7-4.14}$$

For a circular shaft $a = b = r$; then $\tau_{\max} = 2M/\pi r^3$, everywhere on the boundary C.

Displacement Components. With β determined as a function of M and C [Eq. (7-4.6)], the displacement components (u, v) are known for all points in any cross section for a given moment and a given bar. They are $u = -\beta yz$, $v = \beta xz$ [Eq. (7-3.9), with $a = b = 0$]. To compute the displacement component w, we must compute $\psi(x, y)$, the warping function [Eqs. (7-2.2) or (7-3.17)], from its relation to the stress function $\phi(x, y)$ [Eq. (7-3.18)].

By Eqs. (7-3.18) and (7-4.9), we obtain

$$\begin{aligned} \frac{\partial\psi}{\partial x} &= \frac{1}{G\beta}\frac{\partial\phi}{\partial y} + y = \left(1 - \frac{2M}{\pi ab^3 G\beta}\right)y \\ \frac{\partial\psi}{\partial y} &= -\frac{1}{G\beta}\frac{\partial\phi}{\partial x} - x = \left(\frac{2M}{\pi a^3 bG\beta} - 1\right)x \end{aligned} \tag{7-4.15}$$

Integration of Eqs. (7-4.15) yields

$$\psi = \frac{b^2 - a^2}{a^2 + b^2}xy + \text{const} \tag{7-4.16}$$

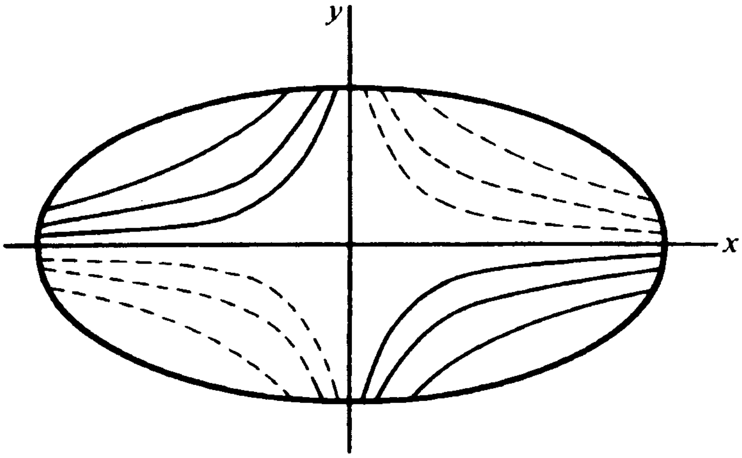


Figure 7-4.3

If we set $w = 0$ for $x = y = 0$, the constant in Eq. (7-4.16) is zero. Consequently,

$$w = \beta\psi = \frac{\beta(b^2 - a^2)}{a^2 + b^2}xy$$

or

$$w = -Kxy \tag{7-4.17}$$

where

$$K = \frac{\beta(a^2 - b^2)}{a^2 + b^2} = \frac{M(a^2 - b^2)}{\pi a^3 b^3 G} \tag{7-4.18}$$

Equation (7-4.17) is the equation of a hyperbola. Accordingly, the contour map of w over the cross section of the bar is represented by a family of hyperbolas (Fig. 7-4.3), with the (x, y) axes representing lines of zero displacement.

Because K is a positive constant, w is positive (in the direction of the positive z axis) in the second and fourth quadrants and negative in the first and third quadrants of the (x, y) plane.

Problem Set 7-4

1. Derive Eq. (7-4.14).
2. Apply the method outlined in Section 7-4 to the bar with circular cross section.

7-5 Remarks on Solutions of the Laplace Equation, $\nabla^2 F = 0$

In the theory of complex variables (Churchill et al., 1989) it is shown that the real and imaginary parts of an analytic function F of the complex variable $z = x + iy$ satisfy the Laplace equation $\nabla^2 F = 0$; that is, the real and imaginary parts of an analytic function are harmonic functions. Accordingly, by considering the real and the imaginary parts of analytic functions F_n, one may proceed, inversely so to speak, to determine the equations of the boundaries of simply connected cross sections for which the real and imaginary parts of F_n represent solutions of the torsion problem. For example, we have previously noted that $f(z) = \psi + i\chi$ is an analytic function where χ is the conjugate harmonic of the warping function ψ, and that the torsion problem may be represented either in terms of ψ or χ (Section 7-3).

One of the simplest sets of analytic functions of the complex variable $z = x + iy$ is the set $F_n = z^n = (x + iy)^n$. By letting $n = \pm 1, \pm 2, \pm 3, \ldots$, solutions of the torsion problem may be developed in the form of polynomials. For example, for $n = 2$, we obtain the solutions $x^2 - y^2$ and $2xy$. For $n = 3$, we find $x^3 - 3xy^2$ and $3x^2y - y^3$. For $n = 4$, we have $x^4 - 6x^2y^2 + y^4$ and $4x^3y - 4xy^3$, and so on. Sums and differences of these polynomial solutions may also be employed, as the sums and the differences of harmonic functions yield other harmonic functions. A systematic application of this technique to the torsion problem has been employed by Weber and Günther (1958). Here we merely present a classical example of the method. Other examples are considered in the problems.

Example 7-5.1. Equilateral Triangle. Consider the harmonic polynomial $\phi_1 = A(x^3 - 3xy^2)$ (obtained from z^n, with $n = 3$), where A is a constant. Because ϕ_1 is harmonic, by setting $\chi = \phi_1$, we may write Prandtl's stress function ϕ in the form [see Eq. (7-3.20)]

$$\phi = -G\beta\left[\frac{(x^2 + y^2)}{2} - \frac{(x^3 - 3xy^2)}{2a} - b\right] \tag{E7-5.1}$$

where a and b denote constants. If we assign the value $2a^2/27$ to the constant b, we may factor Eq. (E7-5.1) into the form

$$\phi = \frac{G\beta}{2a}\left(x - \sqrt{3}y - \frac{2a}{3}\right)\left(x + \sqrt{3}y - \frac{2a}{3}\right)\left(x + \frac{a}{3}\right) \tag{E7-5.2}$$

Accordingly, for $b = 2a^2/27$, the condition that ϕ vanish on the lateral boundary of a bar in torsion [Eqs. (7-3.16)] is satisfied identically by the three

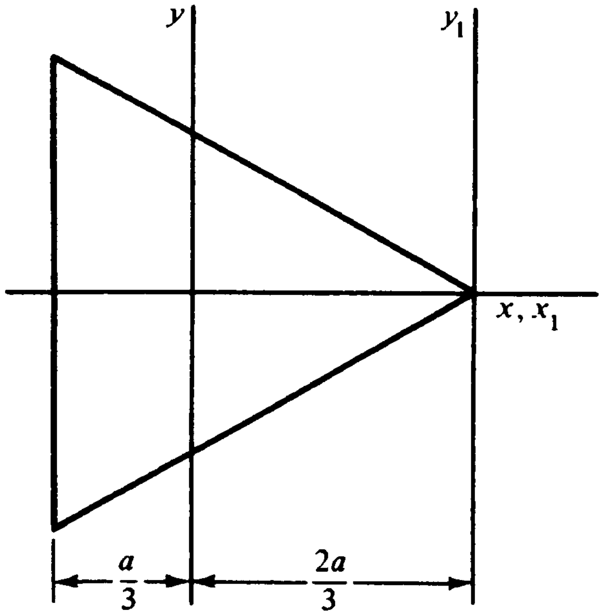


Figure E7-5.1

conditions

$$
\begin{aligned}
x - \sqrt{3}y - \frac{2a}{3} &= 0 \\
x + \sqrt{3}y - \frac{2a}{3} &= 0 \\
x + \frac{a}{3} &= 0
\end{aligned}
\tag{E7-5.3}
$$

Equations (E7-5.3) represent the equations of three straight lines in the (x, y) plane that form an equilateral triangle (Fig. E7-5.1). The region bounded by the three straight lines may be considered as the cross section of a bar in torsion.

Shear–Stress Components. By Eqs. (7-3.3) and (E7-5.1), we find that the shear–stress components are

$$
\begin{aligned}
\tau_{xz} &= -\frac{3G\beta y}{a}\left(x + \frac{a}{3}\right) \\
\tau_{yz} &= G\beta\left(x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right)
\end{aligned}
\tag{E7-5.4}
$$

Equations (E7-5.4) show that $\tau_{xz} = 0$ for $y = 0$ and for $x = -a/3$, and that τ_{yz} is parabolically distributed along the y axis ($x = 0$).

Warping of Cross Section. Letting $\chi = (x^3 - 3xy^2)/2a$ and integrating Eqs. (7-2.14), we obtain the warping function

$$\psi = \frac{y}{2a}(y^2 - 3x^2) + C_0 \tag{E7-5.5}$$

where C_0 is a constant. If we set $w = 0$ for $x = y = 0$, then Eq. (E7-5.5) and the last of Eqs. (7-2.2) yield

$$w = \frac{\beta y}{2a}(y^2 - 3x^2) \tag{E7-5.6}$$

By Eq. (E7-5.6), we note that $w = 0$ for $y = 0$ and $y = \pm\sqrt{3}x$. In general, the w contour lines for which w = constant are described by the equation

$$x^2 = \frac{y^2}{3} + \frac{K}{y} \tag{E7-5.7}$$

where K = constant. If $K > 0$, $x \to \infty$ as $y \to 0$ and as $y \to \infty$. These conditions facilitate the visualization of the contour map for w (Problem 7-5.1), where positive w is taken in the direction of positive z where (x, y, z) for a right-handed coordinate system. The sign of w changes upon crossing the lines $y = 0$ and $y = \pm\sqrt{3}x$. Consequently, the cross section warps into alternate convex ($+w$) and concave ($-w$) regions.

Problem Set 7-5

1. Sketch the contour map for the warping of the triangular cross section under torsion [see Eq. (E7-5.7) and Fig. E7-5.1].
2. Derive Eqs. (E7-5.2), (E7-5.4), and (E7-5.5).
3. Considering terms obtained from the analytic function $(x + iy)^4$, we can express a Prandtl stress function in the form

$$\phi = -G\beta\left[\frac{x^2 + y^2}{2} - \frac{a(x^4 - 6x^2y^2 + y^4)}{2} + \frac{a - 1}{2}\right]$$

Set $a = 0.2$; plot the cross section of the bar for which ϕ solves the torsion problem. Calculate the stress at the boundary point for which the radius vector forms an angle of $\theta = 45°$ with the positive x axis. Use $G = 12 \times 10^6$ psi, $\beta = 0.001$ rad/in. Compare the result to that of a circle with radius equal to the radius vector of the plotted cross section at $\theta = 45°$. Repeat for $a = 0.5$. (In his investigations, Saint-Venant found that the torsional rigidity of a given cross section may be approximated by replacing the given cross section with an elliptical cross section with the same area and the same polar moment of inertia.) Is the *circular* approximation noted above a good approximation?

4. Choosing axes (x_1, y_1) at the tip of the equilateral triangular cross section (Fig. E7-5.1), by means of Eqs. (7-3.6) and (E7-5.2) show that

$$M = \frac{G\beta a^4}{15\sqrt{3}}$$

5. C. Weber proposed the following elementary method of examining the effects of a circular groove or slot in a circular bar [for other kinds of groove and bar combinations, see Weber and Günther (1958)]: Considering a pair of harmonic functions x and $x/(x^2 + y^2)$ obtained from z^n with $n = \pm 1$, Weber transformed the functions into polar coordinates (r, θ). Thus, $x = r\cos\theta$ and $x/(x^2 + y^2) = (\cos\theta)/r$. Hence, he took [see Eq. (7-3.20)] a Prandtl stress function in the form

$$\phi = \frac{G\beta}{2}\left[b^2 - r^2 + 2a(r^2 - b^2)\frac{\cos\theta}{r}\right] \tag{a}$$

where β is taken to denote the angle of twist per unit length. Setting $\phi = 0$ on the boundary, Weber obtained the equation of the boundary of the cross section as

$$(r^2 - b^2)\left(1 - \frac{2a}{r}\cos\theta\right) = 0 \tag{b}$$

Equation (b) is satisfied identically by the conditions

$$\begin{aligned} r^2 - b^2 &= 0 \\ r - 2a\cos\theta &= 0 \end{aligned} \tag{c}$$

Equations (c) may be considered to represent the cross section R of a circular shaft with a circular groove (Fig. P7-5.5). Hence, with Eq. (a), the stress components τ_{xz}, τ_{yz} may be computed by Eqs. (7-3.3). Derive the formulas for τ_{xz}, τ_{yz}.

6. Using the results derived in Problem 5, derive formulas for the stress components τ_{xz}, τ_{yz} on the boundary of the shaft and on the boundary of the groove. Compute the maximum value of stress on the boundary of the shaft; on the groove.

7. Compute τ_{max} in terms of M and a for $\alpha = 60°$, $\alpha = 45°$, and $\alpha = 30°$ (Fig. P7-5.5). Compute τ at the point P for these cases. Verify that $\tau_{xz} = \tau_{yz} = 0$ for corners A and B.

8. For the cross section given in Problem 5, derive the formula for the torsional rigidity of the member.

9. Consider the torsion of a shaft with circular cross section that varies along the axis of the shaft. Let (r, θ, z) be cylindrical coordinates such that (r, θ) lies in the plane of the cross section and z lies along the axis of the shaft. Thus, the radius of the circular cross section varies with z. As in the torsion of a bar with constant circular cross section, assume that $u = w = 0$, where u, w denote displacement components in the (r, z) directions, respectively. Because of the symmetry of the circular cross section, the displacement component v in the θ direction is independent of polar coordinate θ. The dependence of v on r and z is difficult to guess. Hence, take $v = v(r, z)$.

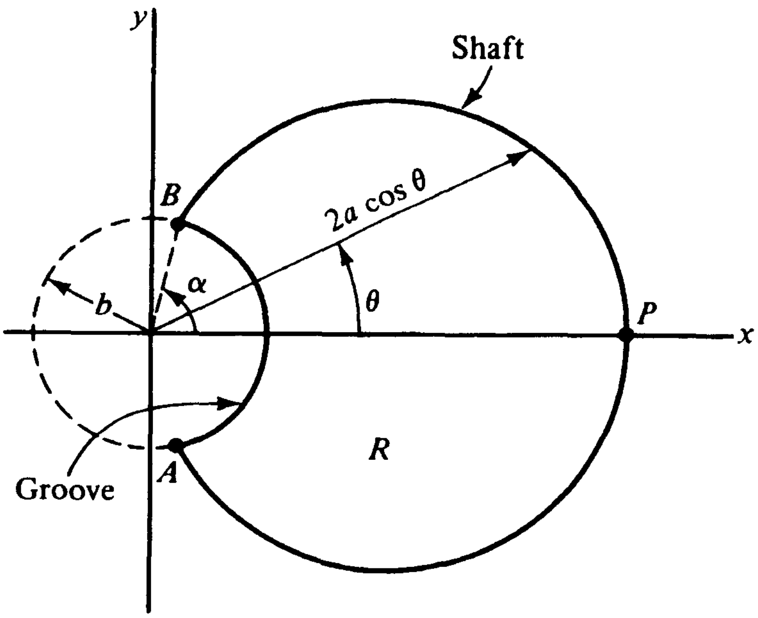


Figure P7-5.5

(a) Determine the corresponding strain components of the shaft.

(b) For a linearly elastic, isotropic material, determine the corresponding stress components of the shaft.

(c) Express the equilibrium equations in terms of v.

(d) Show that there exists a torsion function $F(r, z)$ such that F satisfies the equations of equilibrium, provided

$$\frac{\partial F}{\partial r} = r^3 \frac{\partial}{\partial z}\left(\frac{v}{r}\right), \qquad \frac{\partial F}{\partial z} = -r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right)$$

(e) Show that the defining equation for $F(r, z)$ is

$$\frac{\partial^2 F}{\partial r^2} - \frac{3}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0$$

(f) Determine the boundary conditions that F must satisfy. Hence, define the mathematical problem that determines F. *Hint*: Consider a section of the shaft in the r, z plane and write the boundary conditions for the lateral surface of the shaft.

7-6 Torsion of Bars with Tubular Cavities

Consider a bar with cross section R, where R is the multiply connected region interior to C_0 and exterior to the longitudinal tubular cavities $C_2, C_2, \ldots, C_n$ (Fig. 7-6.1). As in the torsion problem of the simply connected cross section, the displacement components are taken in the form

$$\begin{aligned} u &= -\beta z y \\ v &= \beta z x \\ w &= \beta \psi(x, y) \end{aligned} \tag{7-6.1}$$

where β and ψ are a constant and a function of (x, y), respectively, which are to be determined.

The shearing-stress components in region R are given by the relations [see Eqs. (7-2.4)]

$$\tau_{xz} = \beta G\left(\frac{\partial \psi}{\partial x} - y\right), \qquad \tau_{yz} = \beta G\left(\frac{\partial \psi}{\partial y} + x\right) \tag{7-6.2}$$

Because the boundaries C_0, C_1, $C_2, \ldots, C_n$ are free from external loads, the boundary conditions are

$$l\tau_{xz} + m\tau_{yz} = 0 \qquad \text{on } C_i, \qquad i = 0, 1, \ldots, n \tag{7-6.3}$$

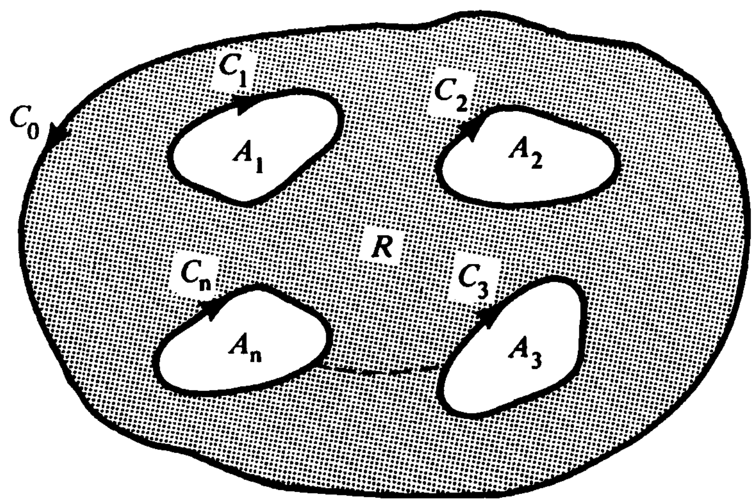


Figure 7-6.1

In terms of ψ, the boundary conditions may be written in the form

$$\frac{d\psi}{dn} = ly - mx \qquad \text{on } C_i, \qquad i = 0, 1, 2, \ldots, n \tag{7-6.4}$$

Introducing the stress function ϕ, defined by Eqs. (7-3.3), we may write the boundary conditions in terms of the stress function ϕ in the form

$$l\frac{\partial\phi}{\partial y} - m\frac{\partial\phi}{\partial x} = \frac{d\phi}{ds} = 0 \qquad \text{on } C_i, \qquad i = 0, 1, \ldots, n \tag{7-6.5}$$

or

$$\phi = K_i \qquad \text{on } C_i, \qquad i = 0, 1, 2, \ldots, n \tag{7-6.6}$$

where the K_i are constants.

In general, the function ϕ may be multiple valued. However, the function ψ is determined by the boundary condition, Eq. (7-6.4), to within an arbitrary constant, and it follows by Eqs. (7-6.2) and (7-3.3) that the function ϕ is determined to within an arbitrary constant. Consequently, the stress function ϕ defined by Eqs. (7-3.3) must satisfy the conditions of Eq. (7-6.6), where the value of only one of the constants K_i may be assigned arbitrarily. If the region R is simply connected (that is, if there are no tubular cavities), $i = 0$, and $\phi = K_0$ on C_0. The constant K_0 may then be assigned an arbitrary value—for example, zero.

The remaining n constants must be chosen so that the displacement component w [and hence ψ, see Eq. (7-6.1)] is a single-valued function, the constants K_i being related to the function ψ through Eqs. (7-3.18) and (7-6.6) or to the complex conjugate χ of ψ through Eqs. (7-3.20) and (7-6.6). For example, the values of K_i may be established so that the solution of the Dirichlet problem [with $b = 0$ in Eqs. (7-3.20)]

$$\begin{aligned} \nabla^2\chi &= 0 \qquad \text{over } R \\ \chi &= \tfrac{1}{2}(x^2 + y^2) + \bar{K}_i, \qquad i = 1, , n \qquad \text{on } C_i \\ G\beta\bar{K}_i &= K_i \end{aligned}$$

satisfies the conditions for the existence of a single-valued function in a multiply connected region.[4]

[4] See Eqs. (7-2.15) and the discussion at the end of Section 5-4 in Chapter 5, particularly Eqs. (5-4.24) and (5-4.25). Here, $m = n$ and $G = \chi$.

Substituting Eqs. (7-3.3) into Eqs. (7-6.2), differentiating the first of Eqs. (7-6.2) by y and the second by x, and subtracting the resulting equations, we obtain the condition

$$\nabla^2 \phi = -2G\beta \qquad \text{in region } R \tag{7-6.7}$$

The twisting moment M that results from the shearing forces that act on the end plane of the bar is

$$M = \iint\limits_{\text{over } R} (x\tau_{yz} - y\tau_{xz})\, dx\, dy \tag{7-6.8}$$

Substituting Eqs. (7-3.3) into Eqs. (7-6.8), we obtain

$$M = -\iint\limits_{\text{over } R} \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right) dx\, dy \tag{7-6.9}$$

Equation (7-6.9) may be written in the form

$$M = \iint\limits_{\text{over } R} 2\phi\, dx\, dy - \iint\limits_{\text{over } R} \left[\frac{\partial(x\phi)}{\partial x} + \frac{\partial(y\phi)}{\partial y} \right] dx\, dy \tag{7-6.10}$$

Transforming the second integral of Eq. (7-6.10) by Green's theorem for the plane (Section 1-16), we may write Eq. (7-6.10) in the form

$$M = 2\iint\limits_{\text{over } R} \phi\, dx\, dy = \sum_{i=0}^{n} \oint_{C_i} \phi(x\, dy - y\, dx) \tag{7-6.11}$$

Because we may assign the value of one of the K_i's in Eq. (7-6.6) arbitrarily, let K_0 on the boundary C_0 be zero; that is, let $\phi = 0$ on C_0. Then, substitution of Eq. (7-6.6) into Eq. (7-6.11) yields

$$M = 2\iint\limits_{\text{over } R} \phi\, dx\, dy + \sum_{i=1}^{n} K_i \oint_{C_i} (y\, dx - x\, dy)$$

Noting that

$$\oint_{C_i} (y\, dx - x\, dy) = 2\iint\limits_{\text{over } A_i} dx\, dy = 2A_i$$

where A_i is the area bounded by the curve C_i, we obtain

$$M = 2\iint\limits_{\text{over } R} \phi\, dx\, dy + 2\sum_{i=1}^{n} K_i A_i \tag{7-6.12}$$

Equation (7-6.12) is the moment–stress function relation for the torsion problem of bars with multiply connected cross sections. Alternatively, by means of Eqs. (7-3.20) and (7-6.12), M may be expressed in terms of the function χ.

Problem Set 7-6

1. For the hollow circular shaft of inner radius a and outer radius b, by the above theory, evaluate M using the stress function $\phi = A(r^2 - b^2)$.

7-7 Transfer of Axis of Twist

In the previous analysis of the torsion problem, we assumed that any cross section of the beam was subjected to an infinitesimal rotation θ about a z axis. No assumption was made as to the location of the z axis relative to the cross section. In calculations, it may be convenient to choose a particular z axis. Hence, let us consider an axis z_1 that is parallel to the axis z, but that intersects the (x, y) plane at point (a, b). With respect to the z_1 axis, the displacement components are

$$u_1 = -\beta z(y - b) \qquad v_1 = \beta z(x - a), \qquad w_1 = \beta\psi_1(x, y) \tag{7-7.1}$$

where ψ_1, not necessarily identical to ψ, is the warping function with respect to the z_1 axis (see also Review Problem R-4, p. 571).

In terms of the stress function ψ_1, the stress components are

$$\begin{aligned} \tau_{xz} &= G\beta\left(\frac{\partial\psi_1}{\partial x} - y + b\right) \\ \tau_{yz} &= G\beta\left(\frac{\partial\psi_1}{\partial y} + x - a\right) \\ \sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \end{aligned} \tag{7-7.2}$$

Substitution of these stress components into the equilibrium equations [Eqs. (7-1.1)] yields the result

$$\nabla^2\psi_1 = \frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = 0 \tag{7-7.3}$$

Also, the boundary conditions [Eqs. (7-1.4)] reduce to the condition

$$\frac{d}{dn}(\psi_1 + bx - ay) = ly - mx \tag{7-7.4}$$

Now the function $\psi_1 + bx - ay$ is harmonic, and it satisfies the same boundary conditions as the warping function ψ. Hence, by the uniqueness (Courant and Hilbert, 1989) of solution of the problem of Neumann, ψ and $\psi_1 + bx - ay$ can differ only by a constant; that is, $\psi_1 = \psi - bx + ay + c$, where c is a constant. Consequently, the displacement components measured with respect to axis z_1 are given by the formulas

$$\begin{aligned} u_1 &= -\beta zy + \beta zb \\ v_1 &= \beta zx - \beta za \\ w_1 &= \beta\psi + \beta ya - \beta xb + \beta c \end{aligned} \qquad (7\text{-}7.5)$$

These components differ by a rigid-body displacement from those with respect to the z axis [Eqs. (7-2.2)]. Consequently, the stress components are identical with those with respect to the z axis. Thus, the choice of the origin of coordinates is immaterial in the torsion problem of the bar with regard to the stress components.

7-8 Shearing–Stress Component in Any Direction

Directional Derivative. Let $P(x, y)$ be any point on a curve in the (x, y) plane. Let the scalar function $\phi(x, y)$ be defined on C with its partial derivatives $\partial\phi/\partial x$ and $\partial\phi/\partial y$; for example, ϕ may be the stress function in torsion. Let Q: $(x + \Delta x, y + \Delta y)$ be a point on C in the neighborhood of P (see Fig. 7-8.1). Let Δs be the length of arc

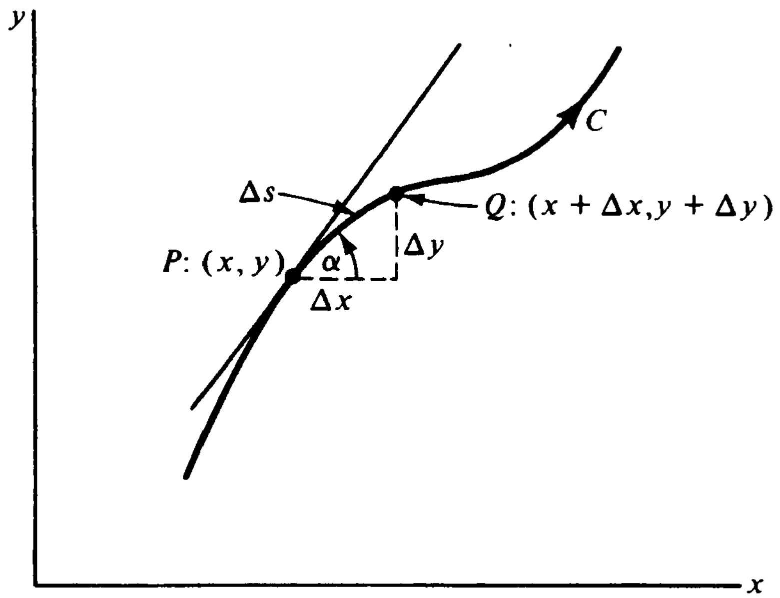


Figure 7-8.1

PQ and $\Delta\phi$ be the change in ϕ due to increments Δx and Δy. Then, the derivative

$$\frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s}$$

determines the rate of change of ϕ along the curve C at the point P: (x, y). Now the total differential of ϕ is

$$d\phi = \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y} dy$$

and

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds}$$

Also,

$$\frac{dx}{ds} = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \cos\alpha$$
$$\frac{dy}{ds} = \lim_{\Delta s \to 0} \frac{\Delta y}{\Delta s} = \sin\alpha$$

Hence, $d\phi/ds = (\partial\phi/\partial x)\cos\alpha + (\partial\phi/\partial y)\sin\alpha$. By this equation, it is apparent that $d\phi/ds$ depends on the direction of s. For this reason, $d\phi/ds$ is called the *directional derivative*. It represents the rate of change of ϕ in the direction of the tangent to the particular curve chosen for point P: (x, y). For example, if $\alpha = 0$,

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}$$

is the rate of change of ϕ in the direction of the x axis.

Maximum Value of the Directional Derivative: Gradient. Consider two neighboring curves in the (x, y) plane; say, C and $C + \Delta C$ (Fig. 7-8.2). Let the respective values of ϕ on these curves be ϕ and $\phi + \Delta\phi$. Then $\Delta\phi/\Delta s$ is the average rate of change of ϕ with respect to the distance Δs measured from curve C to the curve $C + \Delta C$. Now consider the ratio $\Delta n/\Delta s$, where Δn denotes the distance from C to $C + \Delta C$ measured along the normal to C at point P: (x, y). The limiting value of this ratio is $\cos\beta$; that is,

$$\frac{dn}{ds} = \lim_{\Delta C \to 0} \frac{\Delta n}{\Delta s} = \cos\beta$$

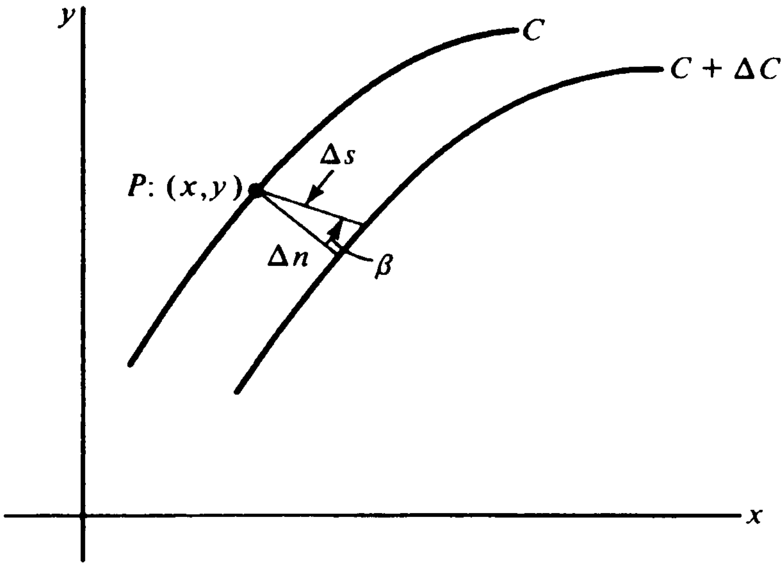


Figure 7-8.2

Hence,

$$\frac{d\phi}{ds} = \frac{d\phi}{dn}\frac{dn}{ds} = \frac{d\phi}{dn}\cos\beta$$

Therefore, $d\phi/dn$, that is, the derivative of ϕ in the direction normal to C, is the maximum value that $d\phi/ds$ may take in any direction. Hence, $(d\phi/ds)_{\text{max}} = |d\phi/dn|$. The vector in the direction of the normal, of magnitude $|d\phi/dn|$, is called the gradient of ϕ; that is, $(\phi_x, \phi_y) = \text{gradient}\ \phi = \text{grad}\,\phi$, where (x, y) subscripts on ϕ denote partial derivatives. Consequently, the maximum value of $d\phi/ds$ is equal to the magnitude of the gradient of ϕ, $|\text{grad}\ \phi|$.

Stress Component–Directional Derivative. Consider an arbitrary point $P: (x, y)$ in the cross section of a bar in torsion (Fig. 7-8.3). The stress component τ_θ in the direction θ is

$$\tau_\theta = \tau_{xz}\cos\theta + \tau_{yz}\sin\theta$$

In terms of the stress function ϕ, by Eqs. (7-3.3),

$$\tau_{xz} = \frac{\partial\phi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial\phi}{\partial x}$$

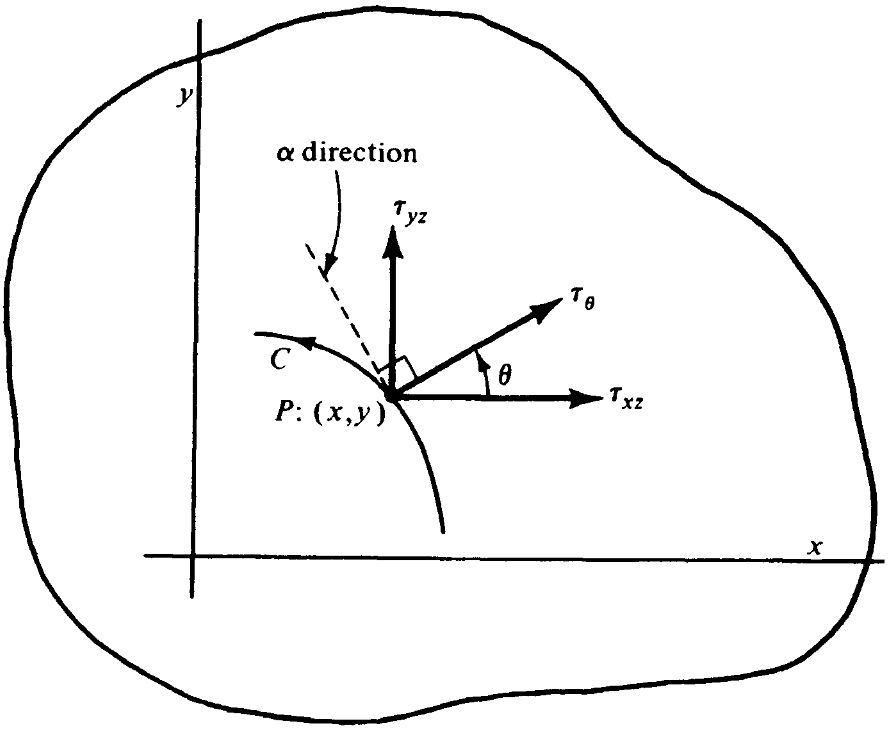


Figure 7-8.3

Therefore,

$$\tau_\theta = \frac{\partial\phi}{\partial y}\cos\theta - \frac{\partial\phi}{\partial x}\sin\theta$$
$$= \phi_y \cos\theta - \phi_x \sin\theta$$

Now set $\alpha = \theta + \pi/2$. Then

$$\tau_\theta = \phi_y \cos\left(\alpha - \frac{\pi}{2}\right) - \phi_x \sin\left(\alpha - \frac{\pi}{2}\right)$$
$$= \phi_x \cos\alpha + \phi_y \sin\alpha = \frac{d\phi}{ds}$$

Consequently, τ_θ is equal to the directional derivative of ϕ in a direction leading θ by 90°. Note that if the direction α corresponds to a direction for which ϕ = constant, $d\phi/ds = 0$. Hence, the shearing stress perpendicular to the line ϕ = constant is zero. Therefore, lines ϕ = constant are shearing–stress trajectories, and the stress vector on lines ϕ = constant has magnitude

$$|\tau_\theta| = \left(\phi_x^2 + \phi_y^2\right)^{1/2} = \left(\frac{d\phi}{ds}\right)_{\max} = |\text{grad}\,\phi|$$

The stress vector is tangent to lines ϕ = constant.

In polar coordinates (r, β) (see Fig. 7-8.4),

$$\tau_r = \frac{1}{r}\frac{\partial \phi}{\partial \beta}, \qquad \tau_\beta = -\frac{\partial \phi}{\partial r} \tag{7-8.1}$$

For example, in terms of polar coordinates (r, β), the Prandtl stress function of a circular shaft with circular groove is [see Eq. (a), Problem 7-5.5]

$$\phi = \frac{G\theta}{2}\left[b^2 - r^2 + 2a(r^2 - b^2)\frac{\cos\beta}{r}\right] \tag{7-8.2}$$

where here θ denotes the unit angle of twist. Consequently, Eqs. (7-8.1) and (7-8.2) yield

$$\begin{aligned} \tau_r = \tau_{rz} &= \frac{1}{r}\frac{\partial \phi}{\partial B} = -\frac{G\theta a}{r^2}(r^2 - b^2)\sin\beta \\ \tau_\beta = \tau_{\beta z} &= -\frac{\partial \phi}{\partial r} = G\theta\left[r - \frac{a}{r^2}(r^2 + b^2)\cos\beta\right] \end{aligned} \tag{7-8.3}$$

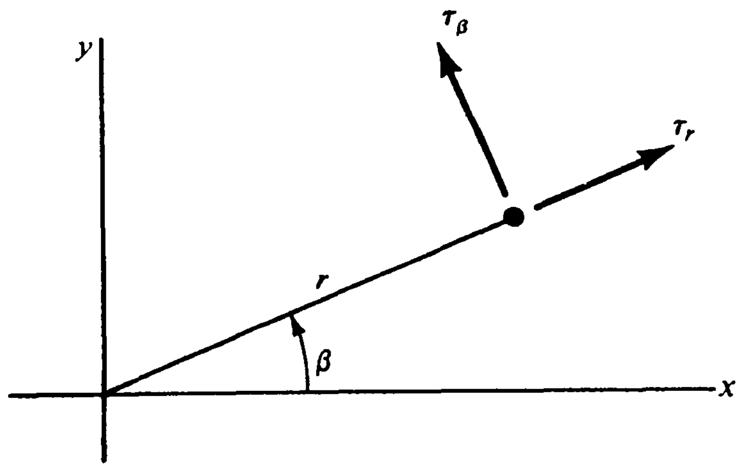


Figure 7-8.4

Thus, for $\beta = 0$ and $r = b$ (Fig. P7-5.5 and Problem 7-5.5) we have

$$\tau_{rz} = 0, \qquad \tau_{\beta z} = -G\theta(2a - b)$$

and for $\beta = 0$ and $r = 2a$ (point P in Fig. P7-5.5) we obtain

$$\tau_{rz} = 0, \qquad \tau_{\beta z} = \frac{G\theta}{4a}(4a^2 - b^2)$$

Problem Set 7-8

1. Plot out several shearing–stress trajectories for the cross section shown in Fig. P7-5.5.

7-9 Solution of Torsion Problem by the Prandtl Membrane Analogy

In this section we consider an analogy method proposed by Prandtl (1903)[5] that leads itself to the obtaining of approximate solutions to the torsion problem. Although this method is of historical interest, it is rarely used today to obtain quantitative results, and it is treated here primarily from the heuristic viewpoint.

The analogy is based upon the equivalence of the torsion equation (7-3.15)

$$\nabla^2\phi = -2G\beta \tag{7-9.1}$$

and the membrane equation

$$\nabla^2 z = -\frac{q}{S} \tag{7-9.2}$$

where z denotes the lateral displacement of a membrane subjected to a lateral pressure q in terms of force per unit area and an initial (large) tension S (Fig. 7-9.1) in terms of force per unit length.

For example, consider an element $ABCD$ of dimensions dx, dy of a membrane (Fig. 7-9.1). The net vertical force due to the tension S acting along edge AD is (assuming small displacements so that $\sin\alpha \approx \tan\alpha$)

$$-S\,dy\sin\alpha \approx -S\,dy\tan\alpha = -S\,dy\frac{\partial z}{\partial x}$$

[5] See L. Prandtl (1903), p. 758. Another analogy method, a hydrodynamic analogy, has been proposed by E. Pestel (1955a, 1955b); see also G. Grossmann (1957). We discuss only the analogy proposed by Prandtl.

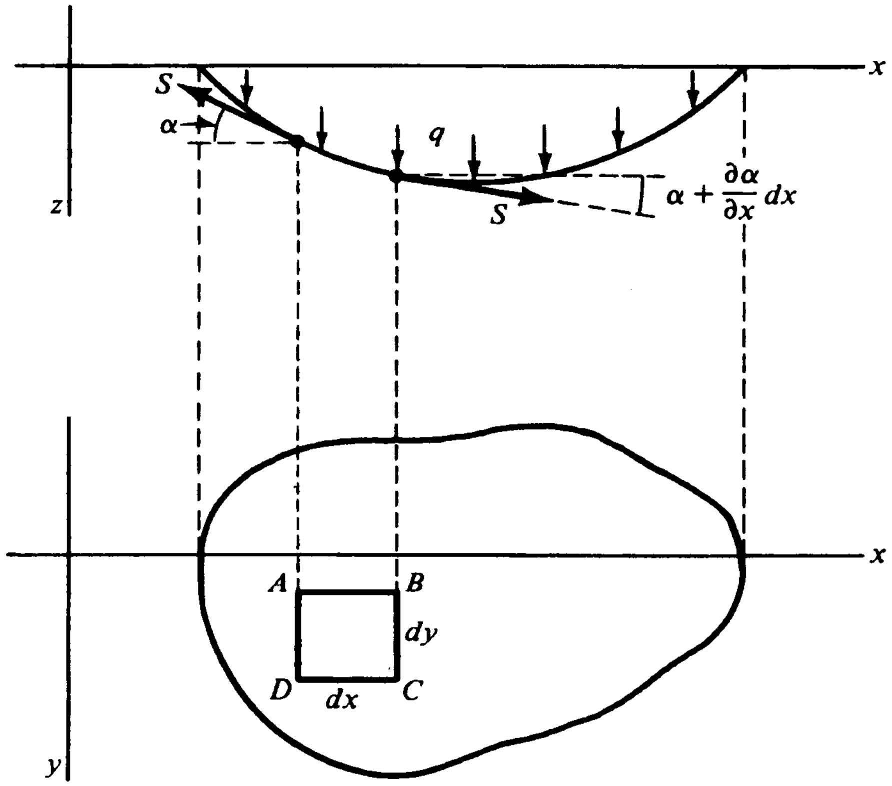


Figure 7-9.1

and similarly the net vertical force due to the tension S (assumed to remain constant for sufficiently small values of q) acting along edge BC is

$$S\,dy \tan\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right) = S\,dy\frac{\partial}{\partial x}\left(z + \frac{\partial z}{\partial x}dx\right)$$

Similarly, for edges AB and DC we obtain

$$-S\,dx\frac{\partial z}{\partial y}, \qquad S\,dx\frac{\partial}{\partial y}\left(z + \frac{\partial z}{\partial y}dy\right)$$

Consequently, summation of force in the vertical direction yields for equilibrium of the membrane element $dx\,dy$

$$S\frac{\partial^2 z}{\partial x^2}dx\,dy + S\frac{\partial^2 z}{\partial y^2}dx\,dy + q\,dx\,dy = 0$$

or

$$\nabla^2 z = -\frac{q}{S}$$

Prandtl showed that the shearing–stress components in a straight elastic bar in torsion may be related to the slopes of a membrane (soap film) extended over a hole in a flat plate and subjected to a small pressure q, the hole having the shape of the cross section of the bar and the membrane being attached to the boundary of the hole.

By comparison of Eqs. (7-9.1) and (7-9.2), we arrive at the following analogous quantities:

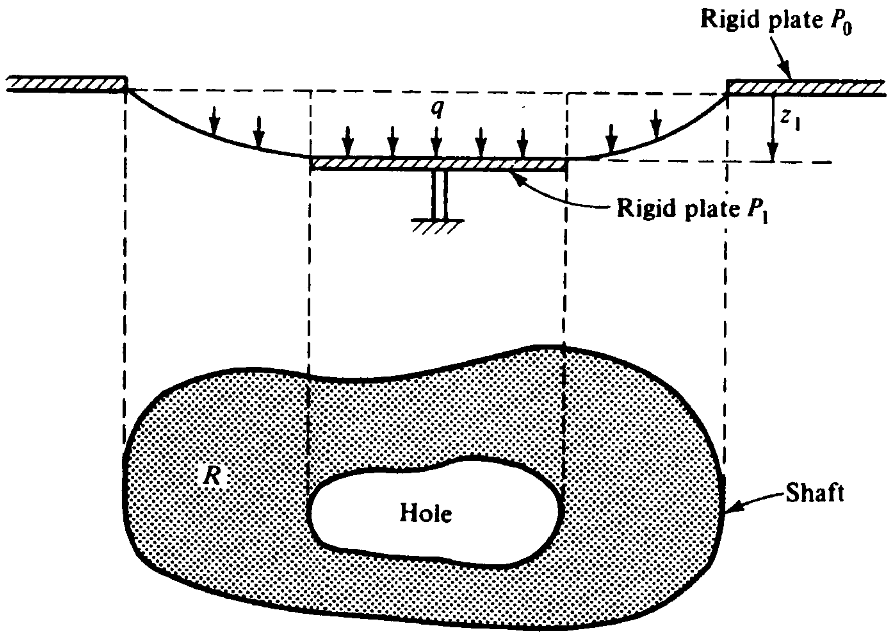
$$z = c\phi, \qquad \frac{q}{S} = c2G\beta \tag{7-9.3}$$

where c is a constant of proportionality. Hence,

$$\frac{z}{q/S} = \frac{\phi}{2G\beta}, \qquad \phi = \frac{2G\beta S}{q} z \tag{7-9.4}$$

Accordingly, the membrane displacement z is proportional to the Prandtl stress function ϕ, and because the shearing–stress components τ_{xz}, τ_{yz} are equal to the appropriate derivatives of ϕ with respect to x and y [see Eqs. (7-3.3)], it follows that the stress components are proportional to the derivatives of the membrane displacement z with respect to the coordinates (x, y) in the flat plate to which the membrane is attached (Fig. 7-9.1). In other words, the stress components at a point (x, y) of the bar are proportional to the slopes of the membrane at the corresponding point (x, y) of the membrane. Consequently, the distribution of shear–stress components in the cross section of the bar is easily visualized by forming a mental image of the slope of the corresponding membrane. Furthermore, for simply connected cross sections, because z is proportional to ϕ, by Eqs. (7-3.6) and (7-9.4) we note that the twisting moment M is proportional to the volume enclosed by the membrane and the (x, y) plane (Fig. 7-9.1).

For the multiply connected cross section, additional conditions arise. For example, consider the cross section shown in Fig. 7-6.1. For this cross section, Eq. (7-6.12) shows that the twisting moment M is proportional to the integral of ϕ over R plus twice the sum of the products of area of the holes and the corresponding constant values of ϕ on the boundaries of the holes. With regard to the membrane analogy, one must then consider a membrane stretched over region R in such a manner that the membrane has a constant value on a boundary of a hole. Such an effect may be obtained if one stretches a membrane over a flat plate P_0 with a cutout corresponding to region R and with flat plates $P_1, P_2, \ldots, P_n$ placed over the holes $A_1, A_2, \ldots, A_n$, the plates $P_1, P_2, \ldots, P_n$ having appropriate heights $z_1, z_2, \ldots, z_n$ with respect to the holes $A_1, A_2, \ldots, A_n$. For example, for a cross section with a single tubular hole, the equivalent membrane is shown in Fig. 7-9.2. This simple



Membrane for plate with single hole

Figure 7-9.2

idea can be extended to n holes. On the basis of the directional derivative concept [see Section 7-8 and particularly Eqs. (7-8.1)] and the membrane analogy, we see that for a curve C on the membrane defined by $z = \text{constant}$ (that is, for $\phi = \text{constant}$) the shear–stress resultant τ is everywhere tangent to the curve (Fig. 7-9.3), where by Eq. (7-8.1),

$$\tau = -\frac{\partial \phi}{\partial n} = -\frac{d\phi}{dn} \qquad \text{on } C \tag{7-9.5}$$

Considering the equilibrium of the part of the membrane enclosed by C, we find

$$qA = \int S \sin\theta \, ds \tag{7-9.6}$$

where A denotes the plane area bounded by C (Fig. 7-9.4).

By Fig. 7-9.4 and Eqs. (7-9.4) and (7-9.5), we have

$$\sin\theta = -\frac{\partial z}{\partial n} = -\frac{d\phi}{dn}\frac{q}{2G\beta S} = \frac{\tau q}{2G\beta S} \tag{7-9.7}$$

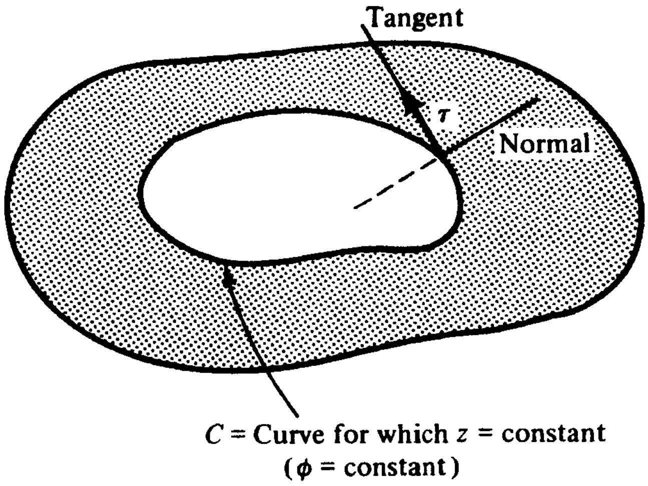


Figure 7-9.3

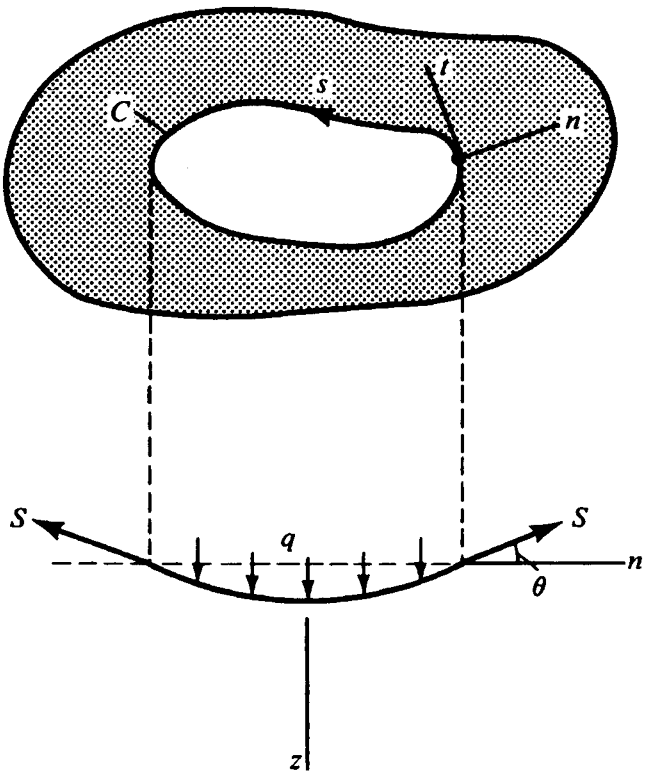


Figure 7-9.4

Hence, Eqs. (7-9.6) and (7-9.7) yield

$$\int_C \tau \, ds = 2G\beta A \tag{7-9.8}$$

Accordingly, for multiply connected regions Eq. (7-9.8) becomes (see Figs. 7-6.1 and 7-9.2)

$$\int_{C_i} \tau \, ds = 2G\beta A_i \tag{7-9.9}$$

where C_i denotes the boundary of the plane area A_i.

Several cross sections and their associated membranes are shown schematically in Fig. 7-9.5.

Some useful conclusions may be drawn from consideration of Fig. 7-9.5. For example, noting that by Eqs. (7-4.6) and (7-6.12)

$$M = 2 \iint_R \phi \, dx \, dy + 2 \sum_{i=1}^{k} K_i A_i = C\beta \tag{7-9.10}$$

it appears from Fig. 7-9.5 that for a bar with circular cross section and a given angle of twist (that is, for a given pressure q and tension S for the associated membrane), the required moment M is not changed as greatly by cutting a concentric circular hole in the shaft as it is cutting a concentric circular hole *and* slit in the shaft (Figs. 7-9.5b, c, and d). Calculations bear out this observation.

Certain kinds of approximations may also be suggested by examination of the membrane. For example, if the wall thickness of a circular tube is small (Fig. 7-9.5b), then by Eq. (7-9.10) we have, with $k = 1$,

$$M = 2 \iint_R \phi \, dx \, dy + 2K_1 A_1 \approx 2K_1 A_1 \tag{7-9.11}$$

where K_1 is the value of ϕ on the boundary of the hole and A_1 is the area of the hole. Other approximations of this type are often employed in practice (Weber and Günther, 1958).

Example 7-9.1. Narrow Rectangular Cross Section. Consider a bar subjected to torsion. Let the cross section of the bar be a solid rectangle with width $2a$ and depth $2b$, where $b \gg a$ (Fig. 7-9.6). The associated membrane is shown in Fig. 7-9.7.

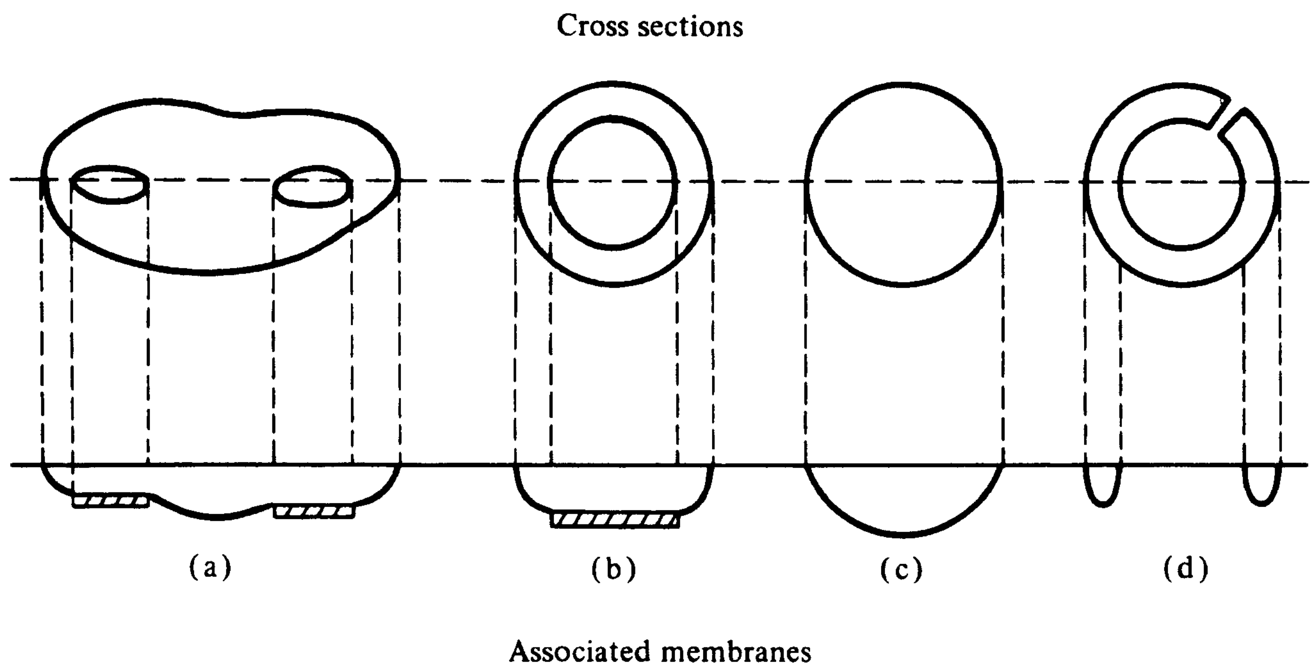


Figure 7-9.5

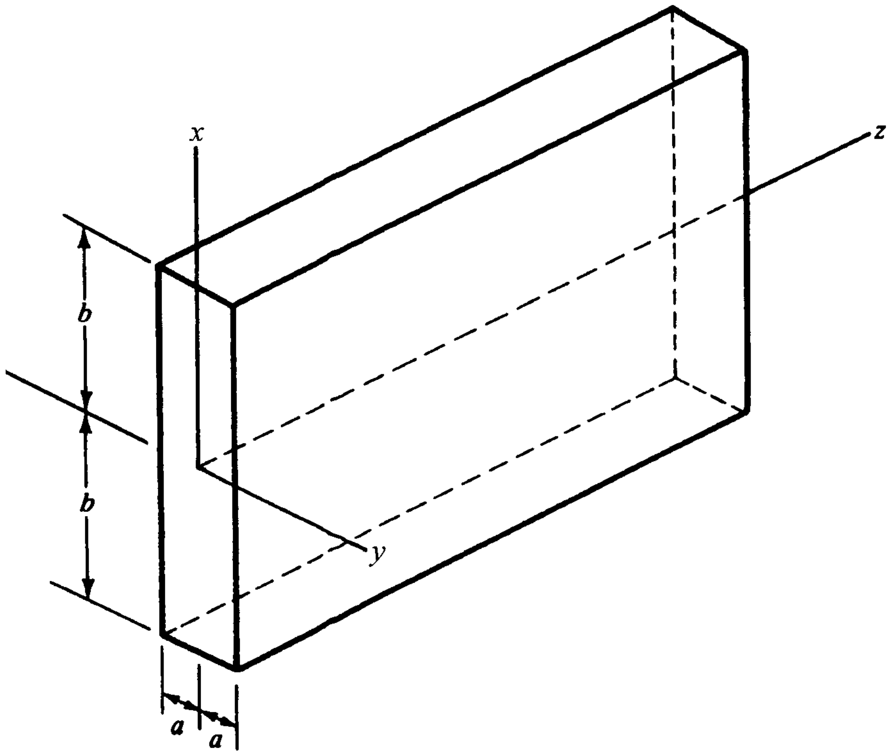


Figure 7-9.6

Except for the region near $x = \pm b$, the membrane deflection is approximately independent of x. For a given x, the deflection with respect to y is assumed to be parabolic. Then

$$z = z_0\left[1 - \left(\frac{y}{a}\right)^2\right] \tag{a}$$

Hence,

$$\nabla^2 z = -\frac{2z_0}{a^2} \tag{b}$$

By Eqs. (b), (7-9.2), and (7-9.3), we may write $\nabla^2 z = -2z_0/a^2 = -2cG\beta$ or

$$\phi = G\beta a^2\left[1 - \left(\frac{y}{a}\right)^2\right] \tag{c}$$

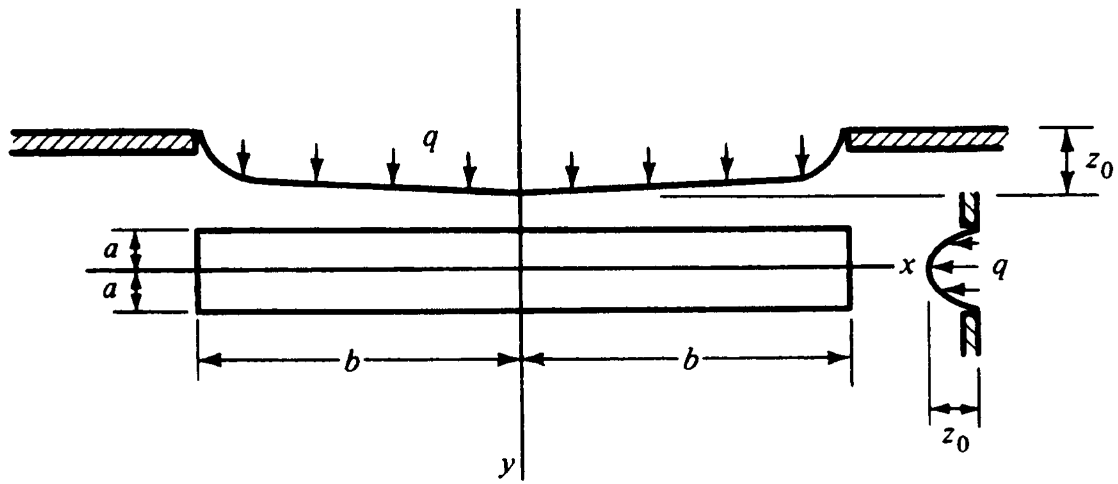


Figure 7-9.7

Consequently, Eqs. (7-3.3) yield

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -2G\beta y, \qquad \tau_{yz} = 0 \tag{d}$$

and the last of Eqs. (7-3.6) yields

$$M = 2\int_{-b}^{b}\int_{-a}^{a} \phi \, dx\, dy = \frac{16}{3}G\beta a^3 b \tag{e}$$

By Eqs. (d), we note that the maximum value of $|\tau_{xz}|$ is $\tau_{\max} = 2G\beta a$ for $y = \pm a$.

In summary, we note that the solution is approximate, and in particular the boundary conditions for $x = \pm b$ are not satisfied. See also Timoshenko (1976) for the case of a narrow trapezoid.

Problem Set 7-9

1. A torsion bar has a cross section in the shape of an isosceles triangle of height h and base $2b$, with $h \gg b$. Let (x, y) axes be defined such that the origin is at the center of the base, with the x axis in the height direction. Define the torsion function to be $\phi = Gb^2\beta[1 - (y/b)^2]$, based upon the membrane of the cross section.
   (a) Derive expressions for the corresponding stress components.
   (b) Determine the formula for the torsional rigidity in terms of G, b, h.
   (c) Examine the boundary conditions and discuss them.

7-10 Solution by Method of Series. Rectangular Section

In Example 7-9.1 the torsion problem of a bar with narrow rectangular cross section was approximated by noting the deflection of the corresponding membrane. In this section we again consider the rectangular section $-a \le x \le a$, $-b \le y \le b$, but we discard the restriction $a \ll b$ (Fig. 7-10.1).

By visualizing the membrane corresponding to the cross section of Fig. 7-10.1, we note that the torsion stress function ϕ must be even in x and y. Also, we recall that in terms of ϕ the torsion problem is defined by the equations

$$\begin{aligned} \nabla^2\phi &= -2G\beta \qquad \text{over } R \\ \phi &= 0 \qquad \text{on } C \end{aligned} \tag{7-10.1}$$

By Example 7-9.1, we have seen that $G\beta(a^2 - x^2)$ is a particular integral of the first of Eqs. (7-10.1). Accordingly, we take the stress function ϕ in the form [see also Eq. (7-3.20)]

$$\phi = G\beta(a^2 - x^2) + V(x, y) \tag{7-10.2}$$

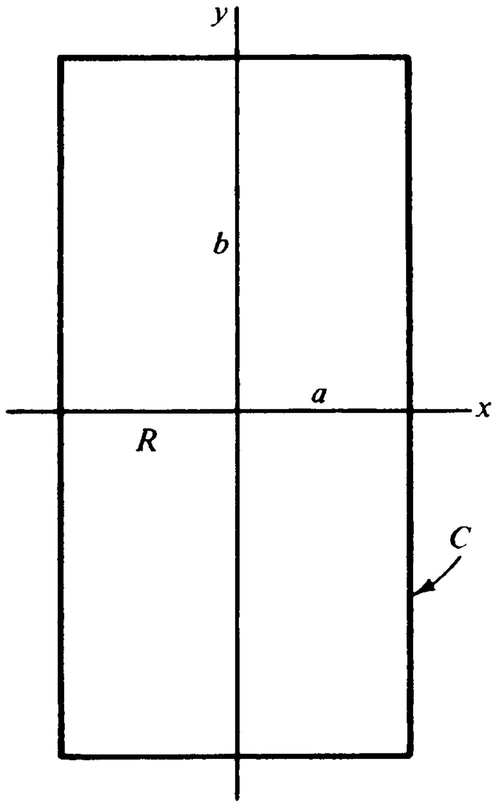


Figure 7-10.1

where $V(x, y)$ is an even function of (x, y). Substitution of Eq. (7-10.2) into Eqs. (7-10.1) yields

$$\begin{aligned} \nabla^2 V &= 0 \qquad \text{over } R \\ V &= 0 \qquad \text{for } x = \pm a \\ V &= G\beta(x^2 - a^2) \qquad \text{for } y = \pm b \end{aligned} \tag{7-10.3}$$

Equations (7-10.3) represent a special case of the Dirichlet problem (Sec. 7-2).

We seek solutions of Eqs. (7-10.3) by the method of separation of variables. Thus, we take

$$V = f(x)g(y) \tag{7-10.4}$$

where $f(x)$ and $g(y)$ are functions of x and y, respectively. The first of Eqs. (7-10.3) and (7-10.4) yield

$$\nabla^2 V = Gf'' + g''f = 0 \tag{7-10.5}$$

where primes denote derivatives with respect to x or y. In order that Eq. (7-10.5) be satisfied, we must have

$$\frac{f''}{f} = -\frac{g''}{g} = -\lambda^2 \tag{7-10.6}$$

where λ^2 is a positive constant. Hence,

$$f'' + \lambda^2 f = 0, \qquad g'' - \lambda^2 g = 0 \tag{7-10.7}$$

The solutions of Eqs. (7-10.7) are

$$\begin{aligned} f &= A\cos\lambda x + B\sin\lambda x \\ g &= C\ \cosh\lambda y + D\sinh\lambda y \end{aligned} \tag{7-10.8}$$

Because V must be even in x and y, it follows that $B = D = 0$. Consequently, the function V takes on the form [Eq. (7-10.4)]

$$V = A\cos\lambda x\cosh\lambda y \tag{7-10.9}$$

where A denotes an arbitrary constant.

To satisfy the second of Eqs. (7-10.3), Eq. (7-10.9) yields the result

$$\lambda = \frac{n\pi}{2a}, \qquad n = 1, 3, 5, \ldots \tag{7-10.10}$$

To satisfy the last of Eqs. (7-10.3) we employ the method of superposition ($\nabla^2 V = 0$ is a linear, homogeneous partial differential equation), and we write

$$V = \sum_{n=1,3,5,\ldots}^{\infty} A_n \cos\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a} \tag{7-10.11}$$

Equation (7-10.11) satisfies $\nabla^2 V = 0$ in R, provided the series converges and is termwise differentiable (Churchill, 1941). Equation (7-10.11) automatically satisfies the boundary condition for $x = \pm a$. The boundary condition for $y = \pm b$ yields the condition [Eqs. (7-10.3)]

$$\sum_{n=1,3,5,\ldots}^{\infty} C_n \cos\frac{n\pi x}{2a} = G\beta(x^2 - a^2) = h(x) \tag{7-10.12}$$

where

$$C_n = A_n \cosh\frac{n\pi b}{2a} \tag{7-10.13}$$

By the theory of Fourier series, we multiply both sides of Eq. (7-10.12) by $\cos(n\pi x/2a)$ and integrate between the limits $-a$ and $+a$ to obtain the coefficients C_n as follows:

$$C_n = \frac{1}{a}\int_{-a}^{a} h(x)\cos\frac{n\pi x}{2a}\,dx \tag{7-10.14}$$

Because $h(x)\cos(n\pi x/2a) = G\beta(x^2 - a^2)\cos(n\pi x/2a)$ is symmetrical about $x = 0$, we may write

$$C_n = \frac{2G\beta}{a}\int_0^a (x^2 - a^2)\cos\frac{n\pi x}{2a}\,dx$$

or

$$C_n = \frac{2G\beta}{a}\int_0^a x^2\cos\frac{n\pi x}{2a}\,dx - 2G\beta a\int_0^a \cos\frac{n\pi x}{2a}\,dx$$

Integration yields (see Pierce and Foster, 1956, Formula 350; or Ryzhik, 1994)

$$C_n = \frac{-32G\beta a^2(-1)^{(n-1)/2}}{n^3\pi^3} \tag{7-10.15}$$

Hence, Eqs. (7-10.11), (7-10.13), and (7-10.15) yield

$$A_n = -\frac{32G\beta a^2(-1)^{(n-1)/2}}{n^3\pi\cosh\dfrac{n\pi b}{2a}} \tag{7-10.16}$$

and

$$\phi = G\beta(a^2 - x^2) - \frac{32G\beta a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\cos\dfrac{n\pi x}{2a}\cosh\dfrac{n\pi y}{2a}}{n^3\cosh\dfrac{n\pi b}{2a}} \tag{7-10.17}$$

Note that as $\cosh x = 1 + x^2/2! + x^4/4! + \cdots$, the series in Eq. (7-10.17) goes to zero if $b/a \to \infty$ (that is, if the section is very narrow $b \gg a$). Then Eq. (7-10.17) reduces to

$$\phi \approx G\beta(a^2 - x^2) \tag{7-10.18}$$

This result verifies the assumption employed in Example 7-9.1 for the slender rectangular cross section.

By Eqs. (7-3.3) and (7-10.17), we obtain

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -\frac{16G\beta a}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n-1)/2} \cos\frac{n\pi x}{2a} \sinh\frac{n\pi y}{2a}}{n^2 \cosh\frac{n\pi b}{2a}}$$
$$\tau_{yz} = -\frac{\partial \phi}{\partial x} = 2G\beta x - \frac{16G\beta a}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n-1)/2} \sin\frac{n\pi x}{2a} \cosh\frac{n\pi y}{2a}}{n^2 \cosh\frac{n\pi b}{2a}} \tag{7-10.19}$$

By Eqs. (7-3.6) and (7-10.17), the twisting moment is

$$M = 2\int_{-b}^{b}\int_{-a}^{a} \phi\, dx\, dy = C\beta = GJ\beta \tag{7-10.20}$$

where J, a factor dependent on geometry of the cross section, is

$$J = 2\int_{-b}^{b}\int_{-a}^{a} (a^2 - x^2)\, dx\, dy - \frac{64a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^3 \cosh\frac{n\pi b}{2a}} \int_{-b}^{b}\int_{-a}^{a} \left(\cos\frac{n\pi x}{2a} \cosh\frac{n\pi y}{2a}\right) dx\, dy$$

Integration yields (see Pierce and Foster, 1956, Formula 489; or Ryzhik, 1994)

$$J = \frac{(2a)^3(2b)}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{a}{b}\right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh\frac{n\pi b}{2a}\right] \tag{7-10.21}$$

The factor outside the brackets on the right side of Eq. (7-10.21) is an approximation for a thin rectangular cross section, because the series goes to zero as b/a becomes large.

In general, Eq. (7-10.21) may be written in the form

$$J = k_1(2a)^3(2b) \tag{7-10.22}$$

where

$$k_1 = \frac{1}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{a}{b}\right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh\frac{n\pi b}{2a}\right] \tag{7-10.23}$$

Equation (7-10.20) may then be written in the form

$$M = G\beta k_1(2a)^3(2b) \tag{7-10.24}$$

Values of k_1 for various ratios of b/a are given by Timoshenko and Goodier (1970).

Problem Set 7-10

1. Verify Eq. (7-10.21).

2. With $b > a$, show that the maximum shear for the rectangular cross section (Fig. 7-10.1) occurs at $x = a$, $y = 0$. Hence, show that

$$\tau_{\max} = 2G\beta a k$$

where

$$k = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2 \cosh(n\pi b/2a)}$$

3. Derive the warping function for the rectangular cross section. Consider the case $a = b$, and sketch in contour lines.

4. Calculate τ_{xz}, τ_{yz} at the indicated points in the cross section (Fig. P7-10.4). Calculate J [Eq. (7-10.21)].

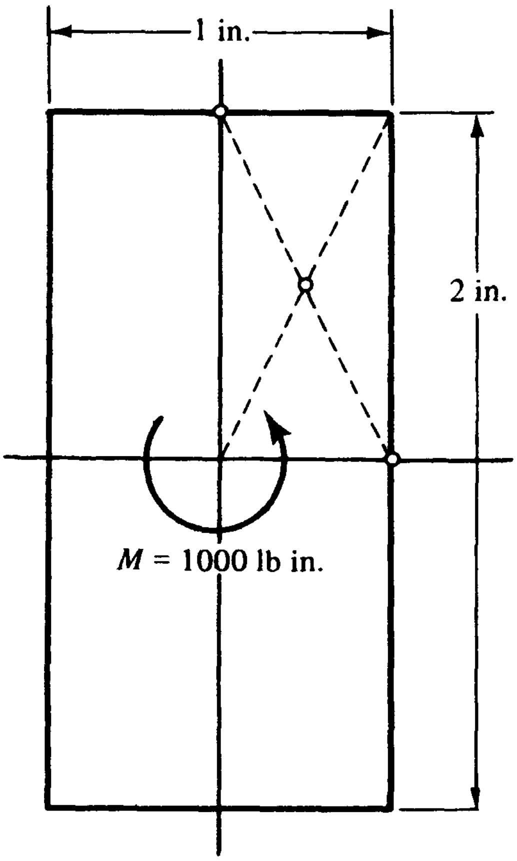


Figure P7-10.4

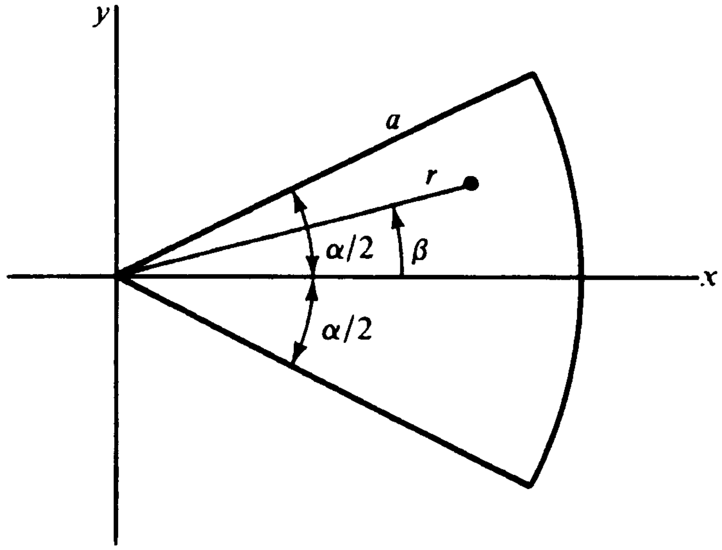


Figure P7-10.5

5. Consider a shaft with a sector cross section with angle α and radius a (Fig. P7-10.5). Let (r, β) denote polar coordinates. Let the torsion stress function ϕ be given by $\phi = V - G\theta r^2/2$, where here θ denotes the unit angle of twist. By the method employed in Section 7-10, show that

$$\phi = \frac{G\theta}{2}\left[-r^2\left(1 - \frac{\cos 2\beta}{\cos\alpha}\right) + \frac{16a^2\alpha^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{(n+1)/2}\left(\frac{r}{a}\right)^{n\pi/\alpha}\frac{\cos(n\pi\beta/\alpha)}{n\left(n+\frac{2\alpha}{\pi}\right)\left(n-\frac{2\alpha}{\pi}\right)}\right]$$

6. Consider the torsion problem of a shaft whose cross section is shown in Fig. P7-10.6 (opposite page). Assume a stress function of the form $\phi = V - \frac{1}{2}G\theta r^2$, where V is a function of r alone, G denotes the shear modulus, θ denotes the angle of twist per unit length of the shaft, and r is the radial polar coordinate. For $h/a \ll 1$, derive an expression for V in terms of a, h, and r. Hence, derive an expression for the shearing stress τ. Discuss the validity of the solution in the vicinity of $\beta = \pi/2$.

7-11 Bending of a Bar Subjected to Transverse End Force

Consider a prismatic elastic bar fixed[6] at the end $z = 0$ and subjected to a lateral force P at the end $z = L$ (Fig. 7-11.1). The cross section of the bar is contained in region R bounded by the surface S. We restrict discussion to the case of simply connected regions R (see Sections 7-2 and 7-6).

[6] For example, the conditions at $z = 0$ may be taken such that the displacement components $u = v = w = 0$ at $x = y = z = 0$, and the rotation $\omega = 0$ at $x = y = z = 0$.

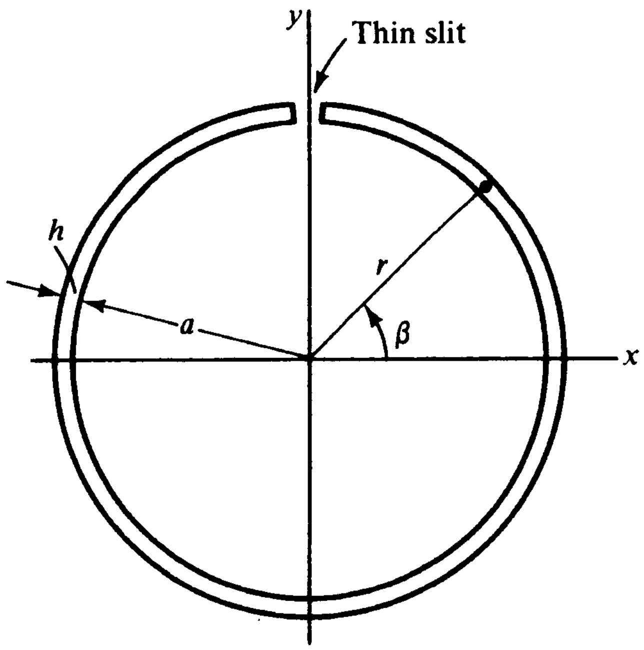


Figure P7-10.6

We let the origin of axes (x, y, z) be located arbitrarily in the cross section at $z = 0$. Furthermore, we take the x axis coincident with the line of action of force P. Then summation of forces on the end face $z = L$ yields

$$P_x = \iint \tau_{zx}\, dx\, dy = P, \qquad P_y = P_z = M_x = M_y = M_z = 0 \tag{7-11.1}$$

Accordingly, overall equilibrium of any portion of the bar (say, between the sections $z = z$, $z = L$, Figs. 7-11.1 and 7-11.2) requires that

$$\begin{aligned} &\iint \tau_{zx}\, dx\, dy = P, \qquad \iint \sigma_z x\, dx\, dy = -P(L - z) \\ &\iint \tau_{zy}\, dx\, dy = \iint \sigma_z\, dx\, dy = \iint y\sigma_z\, dx\, dy \\ &\qquad = \iint (x\tau_{zy} - y\tau_{zx})\, dx\, dy = 0 \end{aligned} \tag{7-11.2}$$

It follows from the first two of Eqs. (7-11.2) that τ_{zx}, and σ_z are not zero. Also, in general, τ_{yz} is not zero by the last of Eqs. (7-11.2).

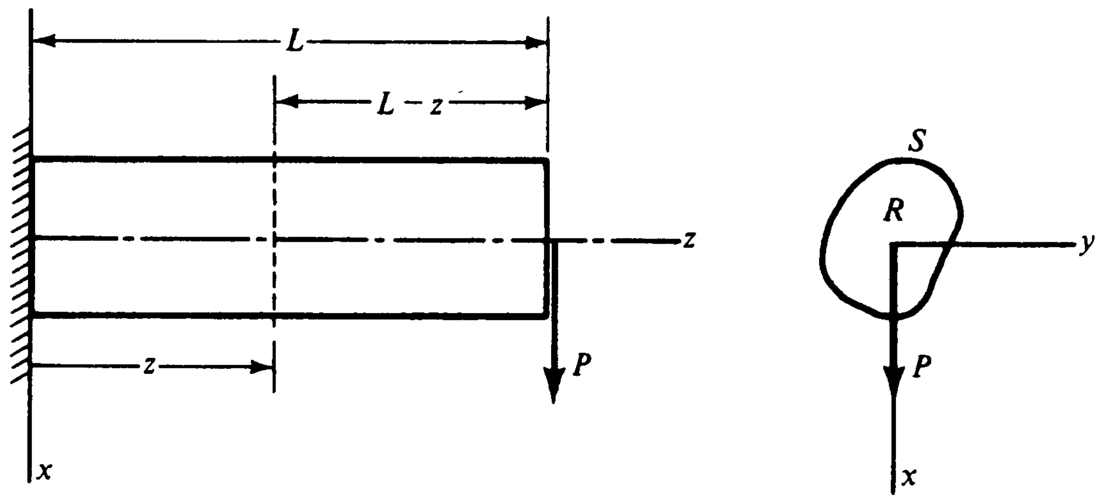


Figure 7-11.1

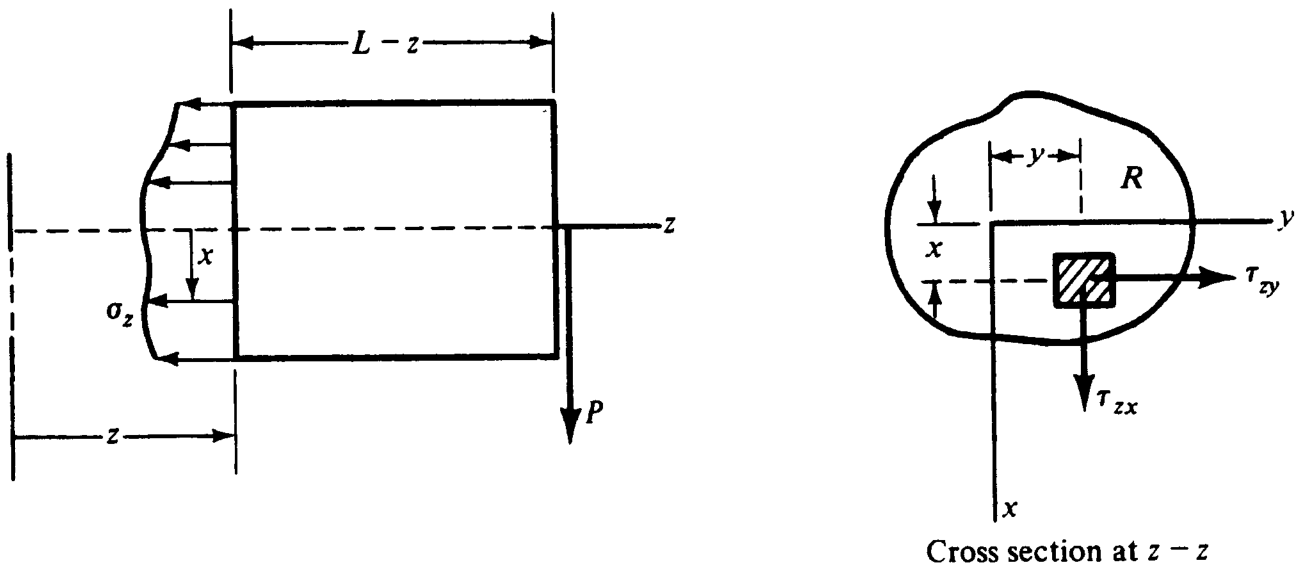


Figure 7-11.2

Following the semi-inverse method of Saint-Venant, we seek solutions such that σ_z, τ_{zx}, τ_{zy} are the only nonvanishing stress components; that is, we assume that

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{7-11.3}$$

Furthermore, we taken the simplest linear dependence on (x, y) for the component σ_z; that is, we assume that σ_z is proportional to $Ax + By + C$, where A, B, C are constants. More explicitly, on the basis of the second of Eqs. (7-11.2) we assume that

$$\sigma_z = P(Ax + By + C)(L - z) \tag{7-11.4}$$

Substitution of Eq. (7-11.4) into Eqs. (7-11.2) yields the result

$$\begin{aligned} AI_{yy} + BI_{xy} + CS_y &= -1 \\ AI_{xy} + BI_{xx} + CS_x &= 0 \\ AS_y + BS_x + CS_0 &= 0 \end{aligned} \tag{7-11.5}$$

where (I_{xx}, I_{yy}, I_{xy}) and (S_x, S_y) are the moments of inertia and the first moments, respectively, of the area of the cross section of the bar relative to axes (x, y), and S_0 is the area of the cross section of the bar.

Equations (7-11.5) are three linear algebraic equations in the unknowns A, B, C. Solving Eqs. (7-11.5), we obtain

$$\begin{aligned} A &= -\frac{I_{xx}S_0 - S_x^2}{\Delta} = \frac{S_x^2 - I_{xx}S_0}{\Delta} \\ B &= \frac{I_{xy}S_0 - S_xS_y}{\Delta} \\ C &= \frac{I_{xx}S_y - I_{xy}S_x}{\Delta} = -A\bar{x} - B\bar{y} \end{aligned} \tag{7-11.6}$$

where

$$\Delta = \begin{vmatrix} I_{yy} & I_{xy} & S_y \\ I_{xy} & I_{xx} & S_x \\ S_y & S_x & S_0 \end{vmatrix} \tag{7-11.7}$$

and $\bar{x}$, $\bar{y}$ denote the coordinates of the center of gravity of the area of the cross section.

In the absence of body forces ($X = Y = Z = 0$). Eqs. (7-1.1), (7-11.3), and (7-11.4) yield

$$\begin{gathered}\frac{\partial \tau_{xz}}{\partial z} = \frac{\partial \tau_{yz}}{\partial z} = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = P(Ax + By + C)\end{gathered} \tag{7-11.8}$$

It follows by the first two of Eqs. (7-11.8) that τ_{xz}, τ_{yz} are independent of z. Furthermore, the last of Eqs. (7-11.8) may be written in the form

$$\frac{\partial}{\partial x}\left[\tau_{xz} - \frac{P}{2}(Ax^2 + Cx)\right] + \frac{\partial}{\partial y}\left[\tau_{yz} - \frac{P}{2}(By^2 + Cy)\right] = 0 \tag{7-11.9}$$

By the theory of Section 1-19, Eq. (7-11.9) represents necessary and sufficient conditions that a function F exist such that

$$\begin{aligned}\tau_{xz} - \frac{P}{2}(Ax^2 + Cx) &= \frac{P}{2}\frac{\partial F}{\partial y} \\ \tau_{yz} - \frac{P}{2}(By^2 + Cy) &= -\frac{P}{2}\frac{\partial F}{\partial x}\end{aligned}$$

or

$$\begin{aligned}\tau_{xz} &= \frac{P}{2}\left[\frac{\partial F}{\partial y} + Ax^2 + Cx\right] \\ \tau_{yz} &= \frac{P}{2}\left[-\frac{\partial F}{\partial x} + By^2 + Cy\right]\end{aligned} \tag{7-11.10}$$

Hence, if τ_{xz} and τ_{yz} are expressed in the form of Eqs. (7-11.10), the equations of equilibrium are satisfied. Furthermore, as τ_{xz} and τ_{yz} are independent of z, it follows that $F = F(x, y)$. The governing equations for F are the compatibility equations [Eqs. (7-1.5)] and the boundary conditions [Eqs. (7-1.4)]. Substitution of Eqs. (7-11.3), (7-11.4), and (7-11.10) into Eqs. (7-1.5) yields

$$\begin{aligned}\frac{\partial}{\partial y}(\nabla^2 F) &= -\frac{2\nu A}{1+\nu} \\ \frac{\partial}{\partial x}(\nabla^2 F) &= \frac{2\nu B}{1+\nu}\end{aligned}$$

Integration yields

$$\nabla^2 F = \frac{2\nu}{1+\nu}(Bx - Ay) - 2C_0 \qquad \text{over } R \tag{7-11.11}$$

where C_0 is a constant of integration that may be interpreted physically [see Section 7-12; see also Eq. (7-11.29)].

The boundary conditions [Eqs. (7-1.4a)] reduce to $l\tau_{xz} + m\tau_{yz} = 0$ or

$$\tau_{xz}\frac{dy}{ds} - \tau_{yz}\frac{dx}{ds} = 0 \tag{7-11.12}$$

where [see Eq. (7-2.7) and Fig. 7-2.2]

$$l = \frac{dy}{ds}, \qquad m = -\frac{dx}{ds} \tag{7-11.13}$$

Substitution of Eqs. (7-11.10) into Eq. (7-11.12) yields

$$\frac{\partial F}{\partial s} = (By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}, \qquad \text{on } S \tag{7-11.14}$$

Equation (7-11.11), which holds over region R, and Eq. (7-11.14), which holds on the lateral surface S, are the defining equations for F.

The above results may be simplified somewhat by noting the nature of Eqs. (7-11.11) and (7-11.14), and representing F in terms of two new functions. Thus, we set

$$F = \Gamma + C_0\phi \tag{7-11.15}$$

Then Eqs. (7-11.11) and (7-11.14) yield

$$\left.\begin{aligned} \nabla^2\phi &= -2 \\ \nabla^2\Gamma &= \frac{2v}{1+v}(Bx - Ay) \end{aligned}\right\} \quad \text{over } R \tag{7-11.16}$$

and

$$\left.\begin{aligned} &\frac{\partial \phi}{\partial s} = 0 \quad \text{or} \quad \phi = 0 \quad \text{(Section 7-6)} \\ &\frac{\partial \Gamma}{\partial s} = (By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds} \end{aligned}\right\} \quad \text{on } S \tag{7-11.17}$$

By Eqs. (7-11.16) and (7-11.17), we see from the theory of Section 7-3 that ϕ (except for the constant factor $G\beta$) is the Prandtl stress function. Accordingly, the problem of the bending of the cantilever bar subjected to transverse end load may be expressed in terms of the Prandtl stress function of torsion and an auxiliary function Γ, which must satisfy the last of Eqs. (7-11.16) and (7-11.17). The function Γ is called the *flexural function* or the *bending function*.

If (x, y) are axes of symmetry with origin at the centroid of the section (then x, y are called principal axes of the cross section),

$$B = C = 0, \qquad A = -\frac{1}{I_{yy}} = -\frac{1}{I} \tag{7-11.18}$$

where I denotes I_{yy}. Then, analogous to the principal-axes theories of stress ($\sigma_{ij} = 0$, $i \neq j$) and strain ($\epsilon_{ij} = 0$, $i \neq j$), axes (x, y) are called principal axes of inertia ($I_{xy} = 0$). With x a principal axis of inertia of the cross section, the equations for the flexural function [Eqs. (7-11.16) and (7-11.17)] reduce to

$$\begin{aligned} \nabla^2 \Gamma &= \frac{2v}{1+v}\frac{y}{I} && \text{over } R \\ \frac{\partial \Gamma}{\partial s} &= \frac{x^2}{I}\frac{dy}{ds} && \text{on } S \end{aligned} \tag{7-11.19}$$

For a certain class of problems it is convenient to redefine Γ in terms of two functions, as follows:

$$\Gamma = \frac{1}{P}[\Psi(x, y) + h(y)] \tag{7-11.20}$$

where Ψ is a function of both x and y, and h is a function of y only.[7] Then Eqs. (7-11.19) become

$$\begin{aligned} \nabla^2 \Psi &= \frac{2v}{1+v}\frac{P}{I}y - \frac{df}{dy} && \text{over } R \\ \frac{\partial \Psi}{\partial s} &= \left(\frac{Px^2}{I} - f\right)\frac{dy}{ds} && \text{on } S \end{aligned} \tag{7-11.21}$$

where $f = dh/dy = f(y)$. The objective of the substitution of Eq. (7-11.20) is to arrive at simpler boundary conditions. For example, if we can choose f such that

$$\left(\frac{Px^2}{I} - f\right)\frac{dy}{ds} = 0 \qquad \text{on } S \tag{7-11.22}$$

then

$$\frac{\partial \Psi}{\partial s} = 0 \qquad \text{on } S \tag{7-11.23}$$

[7] This substitution was employed by S. P. Timoshenko (1913) to solve the problem of flexure of certain kinds of cross sections (Sections 7-14 and 7-15).

and as R is a simply connected region, it follows that we may take (see Section 7-6)

$$\Psi = 0 \qquad \text{on } S \tag{7-11.24}$$

We will employ this technique below to obtain the solution of the flexure problem for the rectangular and the elliptic cross sections.

Alternatively, we may seek solutions of Eq. (7-11.11) by taking a particular integral in the form of a polynomial in x and y. For example, we may express F in the form

$$F = h(x, y) + \frac{v}{3(1+v)}(Bx^3 - Ay^3) - \frac{1}{2}C_0(x^2 + y^2) \tag{7-11.25}$$

where $\nabla^2 h = 0$; that is, h is a harmonic function. Then the problem of bending of a bar by transverse end force transforms into seeking a function such that [see Eqs. (7-11.10), (7-11.11), and (7-11.14)]

$$\begin{aligned} &\nabla^2 h = 0 \qquad \text{over region } R \\ &\frac{\partial h}{\partial s} = (By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds} \\ &\quad + \left(\frac{v}{1+v}Bx^2 - C_0 x\right)\frac{dx}{ds} + \left(\frac{v}{1+v}Ay^2 + C_0 y\right)\frac{dy}{ds} \qquad \text{on } S \end{aligned} \tag{7-11.26}$$

where the stress components are given by

$$\begin{aligned} \tau_{xz} &= \frac{P}{2}\left[\frac{\partial h}{\partial y} + A\left(x^2 - \frac{vy^2}{1+v}\right) + Cx - C_0 y\right] \\ \tau_{yz} &= \frac{P}{2}\left[-\frac{\partial h}{\partial x} + B\left(y^2 - \frac{vx^2}{1+v}\right) + Cy + C_0 x\right] \end{aligned} \tag{7-11.27}$$

For principal axes of the cross section, $B = C = 0$, $A = -1/I$, and Eqs. (7-11.26) and (7-11.27) are simplified accordingly.

Determination of the Constant of Integration, C_0. The above formulation of the flexural problem of the bar (cantilever beam) subjected to end force P is complete except for the determination of the integration constant C_0 [Eq. (7-11.11)]. We find that if we substitute Eqs. (7-11.10) into Eqs. (7-11.2), all the equations are satisfied identically with the exception of the last equation, that is,

$$M_z = \iint (x\tau_{yz} - y\tau_{xz})\, dx\, dy = 0 \tag{7-11.28}$$

The constant C_0 must be chosen to satisfy Eq. (7-11.28). Accordingly, if we employ the definitions of Eqs. (7-11.10) and (7-11.15), we obtain, after some calculations,

$$C_0 \iint \phi \, dx \, dy = -\iint \Gamma \, dx \, dy - \frac{1}{2}\iint (By - Ax)xy \, dx \, dy - \oint \left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dx}{ds} \right] R_s \, ds \tag{7-11.29}$$

where the double integrals are evaluated over R, the line integral is taken over S, and

$$R_s = \frac{1}{2}\int_0^s (x \, dy - y \, dx) \tag{7-11.30}$$

For principal axes of the cross section, $B = C = 0$, $A = -1/I$, and Eq. (7-11.29) is simplified accordingly. With Eq. (7-11.29), the formulation of the problem of bending of a bar subjected to transverse end load is complete.

In general, $C_0 \neq 0$. Hence, there is twisting of the bar (torsion) when a transverse end load is applied arbitrarily. It is for this reason that the Prandtl torsion function [Eqs. (7-11.15) through (7-11.17)] enters into the bending problem of bars.

The constant C_0 may be related to the average rotation of a cross section of the bar with respect to the axis z. For example, for the state of stress defined above, we obtain by Eqs. (7-1.3)

$$\begin{aligned} \epsilon_x = \epsilon_y &= -\frac{\nu\sigma_z}{E} = -\frac{\nu P}{E}(Ax + By + C)(L - z) \\ \epsilon_z &= \frac{P}{E}(Ax + By + C)(L - z) \\ \gamma_{xy} &= 0, \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}, \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \end{aligned} \tag{7-11.31}$$

where

$$\epsilon_x = \frac{\partial u}{\partial x}, \qquad \epsilon_y = \frac{\partial v}{\partial y}, \qquad \epsilon_z = \frac{\partial w}{\partial z}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

and where E is the modulus of elasticity and ν is Poisson's ratio of the material. With Eqs. (7-11.31), the three strain compatibility equations of the type (see Chapter 2, Section 2-16)

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y}$$

are satisfied identically. Also, the equation

$$2\frac{\partial^2 \epsilon_z}{\partial x\, \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

is satisfied. The remaining two equations of compatibility simplify to

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right) &= -\frac{2vPB}{E} \\ \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right) &= \frac{2vPA}{E} \end{aligned} \tag{7-11.32}$$

Integration of Eqs. (7-11.32) leads to

$$\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} = \frac{2vP}{E}(-Bx + Ay) + 2K \tag{7-11.33}$$

where K is a constant of integration.

Recalling the definition of γ_{yz}, γ_{xz}, and ω_z in terms of (u, v, w) [see Eqs. (2-15.14) and (2-5.3)], we note that Eq. (7-11.33) may be written as

$$\frac{\partial \omega_z}{\partial z} = \frac{vP}{E}(-Bx + Ay) + K$$

The term ω_z is the angle of rotation of an element of volume in the rod about the z axis. The term $\partial(\omega_z)/\partial z$ is thus the twist of fibers in the rod parallel to the z axis. Integration of the twist over the cross section R of the bar yields the result

$$\frac{\partial \bar{\omega}_z}{\partial z} = \frac{vP}{E}(-B\bar{x} + A\bar{y}) + K \tag{7-11.34}$$

where

$$\begin{aligned} \bar{\omega}_z &= \frac{1}{S_0}\iint \omega_z\, dx\, dy \\ \bar{x} &= \frac{1}{S_0}\iint x\, dx\, dy \\ \bar{y} &= \frac{1}{S_0}\iint y\, dx\, dy \end{aligned} \tag{7-11.35}$$

denote, respectively, the average value of the angle of rotation ω_z, the x value of the centroid of the cross section, and the y value of the centroid, and S_0 denotes the area of the cross section. Accordingly, by Eq. (7-11.34), the integration constant K may be related to the average angle of rotation of a cross section about the z axis.

Furthermore, if the x axis is an axis of symmetry, $B = \bar{y} = 0$. Then $\partial(\bar{\omega}_z)/\partial z = K$. However, because x is an axis of symmetry (a principal axis passing through the centroid of the cross section), $\bar{\omega}_z = 0$. Hence, when x is a principal axis passing through the centroid of the cross section, $K = 0$.

Alternatively, the compatibility condition, Eq. (7-11.33), may be expressed in terms of τ_{xz} and τ_{yz} by means of the last two of Eqs. (7-11.31). Then, by Eqs. (7-11.10), the compatibility relation may be formulated in terms of the function F. This latter expression, with Eq. (7-11.11), yields the result

$$C_0 = \frac{E}{(1+\nu)P} K \tag{7-11.36}$$

Accordingly, the above remarks made with regard to K hold also for the constant C_0. For example, the constant C_0 defined by Eqs. (7-11.29) vanishes when x is a principal axis. In general, C_0 is related to the mean rotation $\bar{\omega}_z$ by Eqs. (7-11.34) and (7-11.36). That is,

$$\frac{\partial \bar{\omega}_z}{\partial z} = \frac{(1+\nu)P}{E}\left[\frac{\nu}{1+\nu}(-B\bar{x} + A\bar{y}) + C_0\right] \tag{7-11.37}$$

Remark on Solution of $\nabla^2 \chi = F(x, y)$**.** The basic equation of the theories of torsion and of bending of bars is of the form

$$\nabla^2 \chi = F(x, y) \tag{7-11.38}$$

where χ must satisfy certain requirements on the lateral surface of the bar [see, for example, Eqs. (7-2.5), (7-2.10), (7-2.13), (7-2.15), (7-3.10), (7-11.11), (7-11.14), (7-11.16), (7-11.17), (7-11.21), (7-11.26)]. In general, Eq. (7-11.38) is a linear, nonhomogeneous, partial differential equation of second order. Because it is linear, it may be transformed into an equivalent homogeneous equation. The following basic theorem holds for the equivalent homogeneous case ($\nabla^2 \chi = 0$) (Churchill, 1993).

Theorem. *If* $\chi_1, \chi_2, \ldots, \chi_n$ *are n solutions of a homogeneous linear partial differential equation, then* $C_1\chi_1 + C_2\chi_2 + \cdots + C_n\chi_n$ *is also a solution, where* $C_1, C_2, \ldots, C_n$ *are arbitrary constants.*

Any function of x and y that satisfies Eq. (7-11.38) identically is called a *particular* integral. There are in general an infinite number of particular solutions to Eq. (7-11.38). Because of the linear character of Eq. (7-11.38), the sum of a complementary function $(\chi_1, \chi_2, \ldots, \chi_n)$ and *any* particular integral will satisfy Eq. (7-11.38). In the torsion and bending problems of bars, the solution of Eq. (7-11.38) must also satisfy the boundary conditions. In general, the boundary conditions are extremely complex. Particularly, we have seen that in general the bending problem of a bar entails both bending and twisting [see Eqs.(7-11.11), (7-11.15), (7-11.16), and (7-11.17)]. In Section 7-13 we will examine explicitly the conditions under which a bar loaded by a transverse end force will bend without twisting of its end section about the z axis. By application of the conditions for which twisting of the end section is eliminated, we obtain some simplification of the boundary conditions.

Problem Set 7-11

1. Derive Eqs. (7-11.6).
2. Derive Eq. (7-11.29). Simplify the results for principal axes of the cross section.
3. Verify that all but the last of Eqs. (7-11.2) are satisfied by Eq. (7-11.10).

7-12 Displacement of a Cantilever Beam Subjected to Transverse End Force

In this section we derive formulas for the (x, y, z) displacement components (u, v, w) for the stress components defined in Section 7-11. Hence, our task is to integrate Eqs. (7-11.31).

By the third of Eqs. (7-11.31) we have

$$\frac{\partial w}{\partial z} = \frac{P}{E}(Ax + By + C)(L - z)$$

Integration yields

$$w = \frac{PL}{E}(Ax + By + C)z - \frac{P}{2E}(Ax + By + C)z^2 + f(x, y) \tag{7-12.1}$$

where $f(x, y)$ denotes a function of (x, y) only.

To obtain expressions for the displacement components (u, v), we consider simultaneously certain of Eqs. (7-11.31) and Eq. (7-12.1). In the development of these expressions it is convenient to employ the following transformations. As noted by Eqs. (7-11.25) and (7-11.26), the bending problem of the bar may be defined in terms of a harmonic function h. Now we introduce a function $g(x, y)$, the conjugate harmonic of $h(x, y)$, defined by the relations

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}, \qquad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}, \qquad \nabla^2 g = 0 \tag{7-12.2}$$

Then, with Eqs. (7-11.27) and (7-12.2) and the last of Eqs. (7-11.31), we obtain

$$\begin{aligned} \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{(1+v)P}{E}\left[\frac{\partial g}{\partial x} + A\left(x^2 - \frac{vy^2}{1+v}\right) + Cx - C_0 y\right] \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{(1+v)P}{E}\left[\frac{\partial g}{\partial y} + B\left(y^2 - \frac{vx^2}{1+v}\right) + Cy + C_0 x\right] \end{aligned} \tag{7-12.3}$$

By the first of Eqs. (7-11.31) and (7-12.3) and Eq. (7-12.1), we find

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{vP}{E}(Ax + By + C)(L - z) \\ \frac{\partial u}{\partial z} &= \frac{(1+v)P}{E}\left[\frac{\partial g}{\partial x} + A\left(x^2 - \frac{vy^2}{1+v}\right) + Cx - C_0 y\right] \\ &\quad - \frac{PAL}{E}z + \frac{PA}{2E}z^2 - \frac{\partial f}{\partial x} \end{aligned} \tag{7-12.4}$$

Equations (7-12.4) are compatible, provided that

$$\frac{\partial^2(g - \bar{f})}{\partial x^2} = -\frac{(2+v)A}{1+v}x + \frac{vB}{1+v}y - \frac{C}{1+v} \tag{7-12.5}$$

where

$$\bar{f} = \frac{Ef}{(1+v)P} \tag{7-12.6}$$

Similarly, we find

$$\begin{aligned} \frac{\partial v}{\partial y} &= -\frac{vP}{E}(Ax + By + C)(L - z) \\ \frac{\partial v}{\partial z} &= \frac{(1+v)P}{E}\left[\frac{\partial g}{\partial y} + B\left(y^2 - \frac{vx^2}{1+v}\right) + Cy + C_0 x\right] - \frac{PBL}{E}z + \frac{PB}{2E}z^2 - \frac{\partial f}{\partial y} \end{aligned}$$

and

$$\frac{\partial^2(g - \bar{f})}{\partial y^2} = \frac{vA}{1+v}x - \frac{(2+v)B}{1+v}y - \frac{C}{1+v} \tag{7-12.7}$$

Finally, differentiation of the equation

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

with respect to z yields

$$\frac{\partial^2(g-\bar{f})}{\partial x\,\partial y} = \frac{v}{1+v}(Bx+Ay) \tag{7-12.8}$$

Equations (7-12.5), (7-12.7), and (7-12.8) require that [with Eq. (7-12.6)]

$$\begin{aligned} f = {} & \frac{(1+v)P}{E}g + \frac{PA}{2E}\left(\frac{2+v}{3}x^3 - vxy^2\right) \\ & + \frac{PB}{2E}\left(-vx^2y + \frac{2+v}{3}y^3\right) + \frac{PC}{2E}(x^2+y^2) - \beta x + \alpha y + \gamma_0 \end{aligned} \tag{7-12.9}$$

where α, β, γ_0 are constants.

With Eqs. (7-12.1) and (7-12.9), the displacement component w is now determined in terms of the harmonic function g. Next, we substitute the expression for f into the equations for $\partial u/\partial z$ and $\partial v/\partial z$ to obtain [with Eq. (7-11.36)]

$$\begin{aligned} \frac{\partial u}{\partial z} &= -Ky - \frac{P}{E}\left\{A\left[Lz - \frac{z^2}{2} - \frac{v}{2}(x^2-y^2)\right] - vBxy - vCx\right\} + \beta \\ \frac{\partial v}{\partial z} &= Kx - \frac{P}{E}\left\{-vAxy + B\left[Lz - \frac{z^2}{2} + \frac{v}{2}(x^2-y^2)\right] - vCy\right\} - \alpha \end{aligned} \tag{7-12.10}$$

From the equations for $\partial u/\partial x$ and $\partial u/\partial z$, we determine u in the form

$$\begin{aligned} u = {} & -Kyz - \frac{P}{E}\left\{A\left[\frac{Lz^2}{2} - \frac{z^3}{6} + \frac{v}{2}(L-z)x^2 + \frac{v}{2}y^2z\right] + vB(L-z)xy + vC(L-z)x\right\} \\ & + \beta z + F_1(y) \end{aligned}$$

where $F_1(y)$ is an unknown function of y. Similarly, we find

$$\begin{aligned} v = {} & Kxz - \frac{P}{E}\left\{vA(L-z)xy + B\left[\frac{Lz^2}{2} - \frac{z^3}{6} + \frac{v}{2}(L-z)y^2 + \frac{v}{2}x^2z\right] + vC(L-z)y\right\} \\ & - \alpha z + F_2(x) \end{aligned}$$

where $F_2(x)$ is an unknown function of x.

The functions $F_1(y)$ and $F_2(x)$ are determined by the condition

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Hence,

$$F_1(y) = \frac{\nu PAL}{2E} y^2 - \gamma y + \alpha_0$$
$$F_2(x) = \frac{\nu PBL}{2E} x^2 + \gamma x + \beta_0$$

where α_0, β_0, γ are constants.

In summary, by the analysis above we have determined the displacement components (u, v, w) in the form

$$\begin{aligned}
u &= -Kyz - \frac{P}{E}\left\{A\left[\frac{Lz^2}{2} - \frac{z^3}{6} + \frac{\nu}{2}(L-z)(x^2-y^2)\right] + \nu B(L-z)xy + \nu C(L-z)x\right\} \\
&\quad - \gamma y + \beta z + \alpha_0 \\
v &= Kxz - \frac{P}{E}\left\{\nu A(L-z)xy + B\left[\frac{Lz^2}{2} - \frac{z^3}{6} - \frac{\nu}{2}(L-z)(x^2-y^2)\right] + \nu C(L-z)y\right\} \\
&\quad + \gamma x - \alpha z + \beta_0 \\
w &= \bar{g} + \frac{P}{E}\left\{A\left[x\left(Lz - \frac{z^2}{2}\right) + \frac{2+\nu}{6}x^3 - \frac{\nu}{2}xy^2\right]\right. \\
&\quad \left. + B\left[y\left(Lz - \frac{z^2}{2}\right) - \frac{\nu}{2}x^2 y + \frac{2+\nu}{6}y^3\right] + C\left[Lz + \frac{1}{2}(x^2+y^2-z^2)\right]\right\} \\
&\quad - \beta x + \alpha y + \gamma_0
\end{aligned} \tag{7-12.11}$$

where α_0, β_0, γ_0, α, β, γ are constants and

$$\bar{g} = \frac{(1+\nu)Pg}{E} \tag{7-12.12}$$

In Eq. (7-12.11), the terms in α_0, β_0, γ_0, α, β, γ represent a rigid-body displacement (see Chapter 2, Section 2-15 and Problem 2-15.1). To evaluate the rigid-body displacement, we may require that the displacement (u, v, w) and the rotation $(\omega_x, \omega_y, \omega_z)$ be prescribed at a point (x, y, z).

Problem Set 7-12

1. Discuss conditions that may be employed to evaluate α_0, β_0, γ_0, α, β, γ of Eqs. (7-12.11).

7-13 Center of Shear

The condition for which there occurs no twisting of the end section of a bar loaded by transverse end force is defined from Eq. (7-11.37) by settling the twist $\partial(\bar{\omega}_z)/\partial z = 0$. Thus, we obtain

$$C_0 = \frac{v}{1+v}(B\bar{x} - A\bar{y}) \tag{7-13.1}$$

as the necessary and sufficient condition that the twist vanish. In general, if C_0 is defined by Eq. (7-13.1), the moment M_z does not vanish. For example, in general,

$$M_z = \iint (x\tau_{yz} - y\tau_{xz})\,dx\,dy \tag{7-13.2}$$

Accordingly, with Eqs. (7-11.10), (7-11.15), and (7-13.1), Eq. (7-13.2) yields

$$M_z = P\left\{\frac{v}{1+v}(B\bar{x} - A\bar{y})\iint \phi\,dx\,dy + \iint \Gamma\,dx\,dy + \frac{1}{2}\iint (By - Ax)xy\,dx\,dy \right.$$
$$\left. + \oint\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right]R_s\,ds\right\} \tag{7-13.3}$$

Thus, Eq. (7-13.3) defines the moment that must be applied to the end of the bar, together with a force P directed along the x axis, to give zero average twist of the end. By elementary statics and Saint-Venant's principle, we replace the moment M_z and the force P acting along the x axis by a force P_i, parallel to P and equal in magnitude to P, but located at a distance y_i from the x axis, where

$$y_i = -\frac{M_z}{P} = \frac{v}{1+v}(-B\bar{x} + A\bar{y})\iint \phi\,dx\,dy - \iint \Gamma\,dx\,dy - \frac{1}{2}\iint (By - Ax)xy\,dx\,dy$$
$$- \oint\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right]R_s\,ds \tag{7-13.4}$$

The above theory defines bending of a bar with zero average rotation when a force P is applied parallel to the x axis at a distance y_i from the x axis. Similarly, if a force P is applied parallel to the y axis, it must be located at a distance x_i from the y axis for bending of the rod with zero average rotation of the end, where, as with the computation for y_i [Eq. (7-13.4)], we find

$$x_i = \frac{v}{1+v}(b\bar{x} - a\bar{y})\iint \phi\,dx\,dy + \iint \gamma\,dx\,dy + \frac{1}{2}\iint (by - ax)xy\,dx\,dy$$
$$+ \oint\left[(by^2 + cy)\frac{dx}{ds} - (ax^2 + dx)\frac{dy}{ds}\right]R_s\,ds \tag{7-13.5}$$

where over the cross section R

$$\nabla^2 \gamma = \frac{2v}{1+v}(bx - ay) \tag{7-13.6}$$

and on the boundary S

$$\gamma = \oint \left[(by^2 + cy)\frac{dx}{ds} - (ax^2 + cx)\frac{dy}{ds} \right] ds \tag{7-13.7}$$

and where

$$a = \frac{I_{xy}S_0 - S_x S_y}{\Delta}, \qquad b = \frac{S_y^2 - S_0 I_{yy}}{\Delta}, \qquad c = \frac{I_{yy}S_x - I_{xy}S_y}{\Delta} \tag{7-13.8}$$

where Δ is defined by Eq. (7-11.7).

The intersection of the lines $x = x_i$, $y = y_i$ locates a point in the (x, y) plane. This point is called the *shear center*, because if a transverse force is applied at (x_i, y_i), it produces zero average twist of the end of the rod.

It may be shown that the location of the shear center may be determined provided the solution of the torsion problem is known; that is, in general, it is not necessary to know the solution to the bending problem to compute (x_i, y_i) (see Problems 1 through 4 below). In the strength of materials definition of shear center, Poisson's ratio is usually discarded.

Problem Set 7-13

1. With the fact that (Green's theorem)

$$\iint (F\nabla^2 G - G\nabla^2 F)\, dx\, dy = \oint \left(F\frac{\partial G}{\partial n} - G\frac{\partial F}{\partial n} \right) ds \tag{a}$$

where F and G are functions of (x, y), let $F = \phi$, $G = \Gamma$, take into consideration Eqs. (7-11.16) and (7-11.17), and show that

$$\frac{2v}{1+v}\iint \phi(Bx - Ay)\, dx\, dy + 2\iint \Gamma\, dx\, dy = -\oint \Gamma \frac{\partial \phi}{\partial n}\, ds \tag{b}$$

2. Noting by Eqs. (7-2.4) and (7-3.3) that

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} - y, \qquad \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} - x \tag{a}$$

where the factor $G\beta$ has been absorbed in ϕ, show that [with Eqs. (7-2.7), (7-2.9), and (7-11.30)]

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x}\frac{dx}{dn} + \frac{\partial \phi}{\partial y}\frac{dy}{dn} = -\frac{\partial \psi}{\partial s} - 2\frac{dR_s}{ds} \tag{b}$$

Hence, show that

$$\begin{aligned}\oint \Gamma \frac{\partial \phi}{\partial n} ds &= -\oint \Gamma\left(\frac{\partial \psi}{\partial s} + 2\frac{dR_s}{ds}\right) ds \\ &= \oint (\psi + 2R_s)\frac{\partial \Gamma}{\partial s}\, ds \\ &= \oint (\psi + 2R_s)\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] ds \end{aligned} \tag{c}$$

3. With Eqs. (7-2.7) and Eq. (a) of Problem 1, show that

$$\begin{aligned} I &= \oint \psi\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] ds \\ &= -\iint\left\{\frac{\partial}{\partial y}[(By^2 + Cy)\psi] + \frac{\partial}{\partial x}[(Ax^2 + Cx)\psi]\right\} dx\, dy \\ &= -2\iint (Ax + By + C)\psi\, dx\, dy \\ &\quad - \iint\left[(By^2 + Cy)\frac{\partial \psi}{\partial y} + (Ax^2 + Cx)\frac{\partial \psi}{\partial x}\right] dx\, dy \end{aligned}$$

Hence, with Eq. (a) of Problem 2, Eq. (a) of Problem 1, and the fact that $\phi = 0$ on S for a simply connected region R [see Eq. (7-11.17)], show that

$$I = -2\iint (Ax + By + C)\psi\, dx\, dy + \iint (By - Ax)xy\, dx\, dy$$

4. With the results of Problems 1, 2, and 3 above, show that

$$\begin{aligned}\iint \Gamma dx\, dy &= -\frac{v}{1+v}\iint \phi(Bx - Ay)\, dx\, dy \\ &\quad - \oint\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] R_s\, ds \\ &\quad + \iint (Ax + By + C)\psi\, dx\, dy - \frac{1}{2}\iint (By - Ax)\, xy\, dx\, dy \end{aligned}$$

Hence, show that [see Eqs. (7-13.4) and (7-13.5)]

$$
\begin{aligned}
y_i &= -\iint (Ax + By + C)\psi \, dx\, dy + \frac{\nu}{1+\nu}\iint [B(x-\bar{x}) - A(y-\bar{y})]\phi \, dx\, dy \\
x_i &= \iint (ax + by + c)\psi \, dx\, dy + \frac{\nu}{1+\nu}\iint [b(x-\bar{x}) - a(y-\bar{y})]\phi \, dx\, dy
\end{aligned}
\tag{7-13.9}
$$

Equation (7-13.9) shows that if the solution to the torsion problem for region R is known—that is, if either ϕ or ψ is known [see Eq. (a) of Problem 2]—the coordinates (x_i, y_i) of the shear center may be calculated. In other words, (x_i, y_i) may be determined even though the solution of the bending problem ($F = \Gamma + C_0\phi$) is not known.

5. Show that when the cross section of a bar has one axis of symmetry, the shear center will lie on this axis. Show that when the cross section of a bar has two axes of symmetry, the shear center coincides with the intersection of these two axes.

6. Show by calculations and examples that the shear center of a cross section of a bar does not necessarily lie in the region R occupied by the cross section.

7-14 Bending of a Bar with Elliptic Cross Section

In this section we consider a technique introduced by Timoshenko (1921, 1913) for solving the bending problem of bars for certain types of cross section. The motivation of the method lies in seeking to represent the boundary conditions [taken in the form of the second of Eqs. (7-11.21)] in the simplest possible form. For example, for a simply connected cross section we may choose $f(y)$ to make the right side of the second of Eqs. (7-11.21) equal to zero. Then, $\partial\Psi/\partial s = 0$ on S. Because the cross section is simply connected, it follows that Ψ may be taken equal to zero on S (see Section 7-6). We illustrate the method for a bar with elliptic cross section.

For an elliptic cross section, the lateral surface of the cross section is defined by the equation

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7-14.1}
$$

where (a, b) denotes the major and minor semiaxes of the ellipse. Accordingly, the right side of the second of Eqs. (7-11.21) vanishes identically, provided we set

$$
f(y) = -\frac{Pa^2}{Ib^2}(y^2 - b^2) \tag{7-14.2}
$$

Substitution of Eq. (7-14.2) into the first of Eqs. (7-11.21) yields

$$
\nabla^2\Psi = \frac{2Py}{I}\left(\frac{a^2}{b^2} + \frac{\nu}{1+\nu}\right) \tag{7-14.3}
$$

The boundary condition $\Psi = 0$ on S [see Eq. (7-11.24)] will be satisfied if we take Ψ in the form

$$\Psi(x, y) = D\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)y \tag{7-14.4}$$

where $D =$ constant. Substitution of Eq. (7-14.4) into Eq. (7-14.3) yields

$$D = \frac{P}{I}\left(\frac{a^2b^2}{3a^2 + b^2}\right)\left(\frac{a^2}{b^2} + \frac{\nu}{1+\nu}\right) \tag{7-14.5}$$

With the cross section defined by Eq. (7-14.1), the axes (x, y) are axes of symmetry. Hence, with the resultant force P directed along the x axis, $C_0 = 0$ (see Section 7-11). Then Eqs. (7-11.10), (7-11.15), (7-11.20), and (7-11.21) yield the following expressions for the stress components τ_{xz}, τ_{yz}:

$$\begin{aligned} \tau_{xz} &= \frac{1}{2}\left[\frac{\partial \Psi}{\partial y} + f - \frac{Px^2}{I}\right] \\ \tau_{yz} &= -\frac{1}{2}\frac{\partial \Psi}{\partial x} \end{aligned} \tag{7-14.6}$$

Substitution of Eqs. (7-14.2), (7-14.4), and (7-14.5) into Eqs. (7-14.6) yields

$$\begin{aligned} \tau_{xz} &= \frac{Pa^2}{2I}\left[\frac{(1+\nu)a^2 + \nu b^2}{(1+\nu)(3a^2+b^2)}\left(\frac{x^2}{a^2} + \frac{3y^2}{b^2} - 1\right) - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\right] \\ \tau_{yz} &= -\frac{(1+\nu)a^2 + \nu b^2}{(1+\nu)(3a^2+b^2)}\frac{Pxy}{I} \end{aligned} \tag{7-14.7}$$

The normal stress component σ_z for this case ($A = -1/I, B = C = 0$) is found from Eq. (7-11.4) to be

$$\sigma_z = -\frac{Px}{I}(L - z) \tag{7-14.8}$$

Equation (7-14.8) agrees precisely with elementary beam theory. However, the shearing–stress components differ from results predicted by elementary beam theory. Elementary beam theory predicts that τ_{yz} vanishes everywhere and that τ_{xz} is a function of x only.

If $b \ll a$, Eqs. (7-14.7) may be approximated by the equations

$$\tau_{xz} = \frac{P}{3I}(a^2 - x^2), \qquad \tau_{yz} = -\frac{Pxy}{3I} \tag{7-14.9}$$

Then, τ_{xz} agrees with the stress component computed by elementary theory. However, again τ_{yz} is in disagreement with elementary theory although it is very small (because y is small; it is at most equal to b). The maximum value of τ_{xz} predicted by Eq. (7-14.9) is (for $x = 0$)

$$(\tau_{xz})_{\max} = \frac{Pa^2}{3I} = \frac{4P}{3A} \tag{7-14.10}$$

where $I = Aa^2/4$, where A = cross-sectional area of the ellipse.

By Eqs. (7-14.7), the maximum value of τ_{xz} is (for $x = y = 0$)

$$(\tau_{xz})_{\max} = \frac{Pa^2}{2I}\left[1 - \frac{(1+\nu)a^2 + \nu b^2}{(1+\nu)(3a^2+b^2)}\right] \tag{7-14.11}$$

Again, for $b \ll a$, Eq. (7-14.11) yields the result given by Eq. (7-14.10).

Bar with Circular Cross Section. If in the above analysis we let $a = b$, the cross section of the bar becomes circular. Thus, for the circular bar we obtain from Eqs. (7-14.7)

$$\begin{aligned} \tau_{xz} &= \frac{P}{2I}\left[\frac{1+2\nu}{4(1+\nu)}(x^2+3y^2-a^2) - (x^2+y^2-a^2)\right] \\ \tau_{yz} &= -\frac{1+2\nu}{4(1+\nu)}\frac{Pxy}{I} \end{aligned} \tag{7-14.12}$$

Hence

$$(\tau_{xz})_{\max} = \frac{3+2\nu}{8(1+\nu)}\frac{Pa^2}{I} \tag{7-14.13}$$

7-15 Bending of a Bar with Rectangular Cross Section

Consider a cantilever beam with rectangular cross section R and with lateral surface S. Let end load P be applied to the end of the bar (beam) and directed along the vertical centroidal axis (x axis, Fig. 7-15.1). The cross section is defined by the equation

$$(x^2 - a^2)(y^2 - b^2) = 0 \tag{7-15.1}$$

By the theory of Section 7-11, the beam undergoes bending with no twisting of the end plane ($A = -1/I$, $B = C = C_0 = 0$).

Because the net load P is equivalent to shear–stress components τ_{xz}, τ_{yz} distributed over the end of the bar, we may employ the semi-inverse method by assuming simple distributions for τ_{xz}, τ_{yz} and then attempt to satisfy the elasticity equations. For example, as $\sum F_x = P$, $\sum F_y = 0$, it appears reasonable to assume τ_{yz} to be odd in y and τ_{xz} to be even in x and y (see Fig. 7-15.1). Furthermore, we

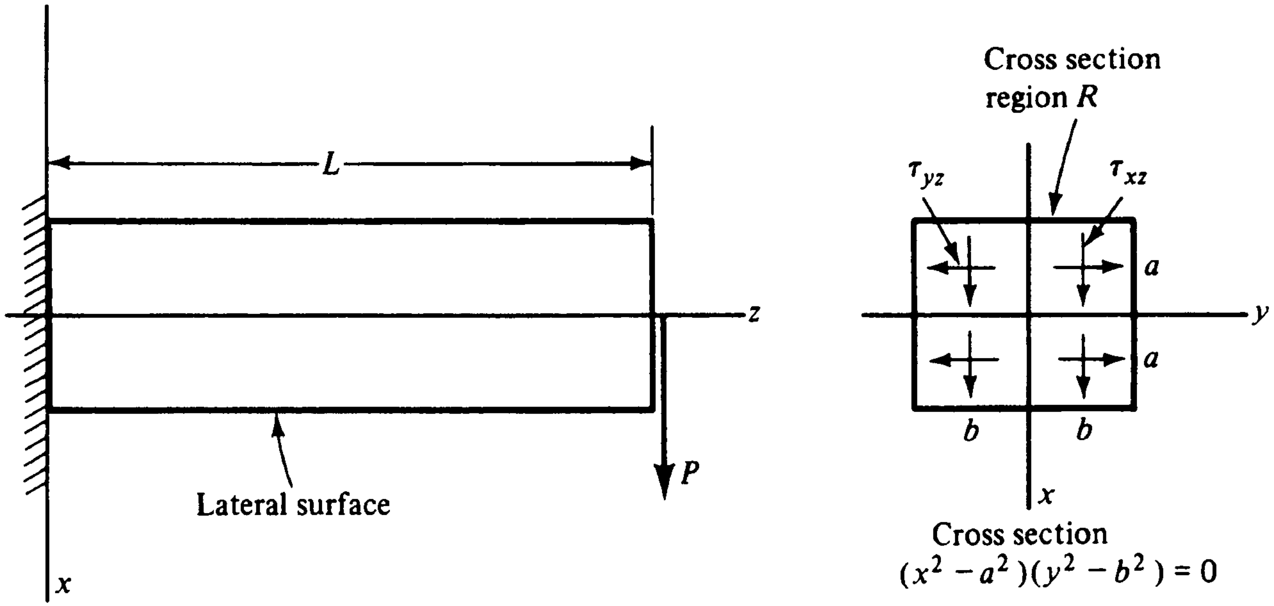

L
z
x
P
Lateral surface
Cross section
region R
τ_{yz}
τ_{xz}
a
a
b
b
y
x
Cross section
$(x^2 - a^2)(y^2 - b^2) = 0$

Figure 7-15.1

employ the technique demonstrated in Section 7-14 for the elliptic cross section. Hence, taking $f = Pa^2/I$ and noting the nature of the dependency of (τ_{xz}, τ_{yz}) on x and y, we find by Eqs. (7-14.6) that Ψ is even in x and odd in y. Also, by choosing $f = Pa^2/I$ and noting that $dy/ds = 0$ for $y = \pm b$, by Eqs. (7-11.21), we obtain

$$\begin{aligned} \nabla^2\Psi &= \frac{2\nu}{1+\nu}\frac{Py}{I} \qquad \text{over } R \\ \Psi &= 0 \qquad \text{on } S \end{aligned} \tag{7-15.2}$$

By inspection, a particular solution of the first of Eqs. (7-15.2) is

$$\Psi_1 = Ay^3 + By \tag{7-15.3}$$

Substitution of Eq. (7-15.3) into Eq. (7-15.2) yields

$$A = \frac{\nu P}{3(1+\nu)I} \tag{7-15.4}$$

with B arbitrary.

By the discussion at the end of Section 7-11, we choose Ψ in the form

$$\Psi = \Omega + \frac{\nu P}{3(1+\nu)I}y^3 + By \tag{7-15.5}$$

where by Eqs. (7-15.2) and (7-15.5)

$$\begin{aligned} \nabla^2\Omega &= 0 \qquad \text{on } R \\ \Omega &= -\frac{\nu P}{3(1+\nu)I}y^3 - By \qquad \text{on } S \end{aligned} \tag{7-15.6}$$

Let us choose[8] B so that $\Omega = 0$ for $y = \pm b$. Then, Eq. (7-15.6) yields

$$B = -\frac{\nu P b^2}{3(1+\nu)I} \tag{7-15.7}$$

Thus, by Eqs. (7-15.5) and (7-15.6), we arrive at the stress function

$$\Psi = \Omega + \frac{\nu P}{3(1+\nu)I}(y^3 - b^2 y) \tag{7-15.8}$$

[8] Note that we could assume a particular solution of Eq. (7-15.2) in the form $Ay^3 + B$. Then we could choose B so that $\Omega = 0$ for $y = b$, but Ω would not be zero on the line $y = -b$. Hence, our choice of Ψ_1 [Eq. (7-15.3)] leads to a simpler boundary condition for Ω.

where

$$\nabla^2\Omega = 0 \qquad \text{on } R \tag{7-15.9}$$

and

$$\begin{aligned} \Omega &= 0 \qquad \text{for } y = \pm b \\ \Omega &= \frac{\nu P}{3(1+\nu)I}(b^2 y - y^3) \qquad \text{for } x = \pm a \end{aligned} \tag{7-15.10}$$

Because Ψ is even in x and odd in y, Ω is even in x and odd in y.

Consider solutions of Eq. (7-15.9) of the form

$$\Omega = f(x)g(y) \tag{7-15.11}$$

where $f(x)$, $g(y)$ are functions of x and y, respectively. Substitution of Eq. (7-15.11) into Eq. (7-15.9) with the requirement that Ω be even in x and odd in y yields solutions of the form.

$$\Omega = A \cosh kx \sin ky \tag{7-15.12}$$

where A and k are constants.

Substitution of Eq. (7-15.12) into the first of Eqs. (7-15.10) yields $A \cosh kx \sin kb = 0$, or $k = n\pi/b$, $n = 1, 2, 3, \ldots$. Hence, superposition of solutions of the type given by Eq. (7-15.12) yields

$$\Omega = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \tag{7-15.13}$$

Let $A_n \cosh(n\pi a/b) = a_n$. Then, by Eq. (7-15.13) and the second of Eqs. (7-15.10), we must require that

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} = \frac{\nu P}{3(1+\nu)I}(b^2 y - y^3) \tag{7-15.14}$$

Multiplying Eq. (7-15.14) by $\sin(m\pi y/b)$ and integrating from $-b$ to b, we obtain

$$\sum_{n=1}^{\infty} a_n \int_{-b}^{b} \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b}\, dy = \frac{\nu P}{3(1+\nu)I} \int_{-b}^{b} (b^2 y - y^3) \sin \frac{m\pi y}{b}\, dy \tag{7-15.15}$$

Observing that

$$\int_{-b}^{b} \sin\frac{m\pi y}{b}\sin\frac{n\pi y}{b}\,dy = \begin{cases} 0, & m \neq n \\ b, & m = n \end{cases}$$

$$\int_{-b}^{b} y\sin\frac{m\pi y}{b}\,dy = -\frac{2(-1)^m b^2}{m\pi}.$$

$$\int_{-b}^{b} y^3\sin\frac{m\pi y}{b}\,dy = -\frac{2(-1)^m b^4}{m^3\pi^3}(m^2\pi^2 - 6)$$

we obtain after integration of Eq. (7-15.15)

$$a_n = -\frac{4\nu P b^3}{(1+\nu)I}\frac{(-1)^n}{n^3\pi^3}$$

Hence, the constant A_n in Eq. (7-15.13) is determined, and the stress function Ψ is given by the formula

$$\Psi = \frac{\nu P}{3(1+\nu)I}\left[y^3 - b^2 y - \frac{12b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\cosh\dfrac{n\pi x}{b}\sin\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right] \tag{7-15.16}$$

Then, because $f = Pa^2/I$, substitution of Eq. (7-15.16) into Eqs. (7-14.6) yields

$$\tau_{xz} = \frac{P}{2I}(a^2 - x^2) + \frac{\nu P}{6(1+\nu)I}\left[3y^2 - b^2 - \frac{12b^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cosh\dfrac{n\pi x}{b}\cos\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right]$$

$$\tau_{yz} = \frac{2\nu P b^2}{\pi^2(1+\nu)I}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\sinh\dfrac{n\pi x}{b}\sin\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}} \tag{7-15.17}$$

Equations (7-15.17) express the solution to the bending of a cantilever beam with rectangular cross section and with load P directed along the vertical centroidal axis in the end plane $z = L$.

Examination of τ_{xz}. On the horizontal line $x = 0$, Eqs. (7-15.17) yield

$$\tau_{xz} = \frac{Pa^2}{2I}\left\{1 + \frac{\nu}{1+\nu}\left[\frac{y^2}{a^2} - \frac{b^2}{3a^2} - \frac{4b^2}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cos\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right]\right\} \tag{7-15.18}$$

$$\tau_{yz} = 0$$

Elementary theory of beams yields the result (for $x = 0$) $\tau_{xz} = Pa^2/2I$. Hence, the quantity in braces in Eq. (7-15.18) represents a correction factor K to elementary beam theory; that is, the result of elementary beam theory must be multiplied by the factor

$$K = 1 + \frac{v}{1+v}\left[\frac{y^2}{a^2} - \frac{b^2}{3a^2} - \frac{4b^2}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cos\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right] \tag{7-15.19}$$

If $v = 0$, the correction factor is 1. Also, if $b \ll a$ the correction factor is approximately 1. That is, elementary theory is approximately correct (at $x = 0$) for beams of narrow cross section (Fig. 7-15.1, with $b \ll a$). The fact that the correction factor approaches 1 as b/a approaches zero is apparent from Eq. (7-15.19), as $\cos(n\pi y/b)$ is never larger than 1 and $\cosh(n\pi a/b)$ becomes very large as $b/a \to 0$.

It may also be shown that $K \to 1$ as b/a becomes very large: For example, consider the point $x = y = 0$. Then, Eq. (7-15.19) yields

$$K = 1 - \frac{v}{1+v}\left[\frac{b^2}{3a^2} + \frac{4b^2}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\operatorname{sech}\frac{n\pi a}{b}\right]$$

Note that as $b/a \to \infty$, sech $(n\pi a/b) \to 1$. To evaluate the series $\sum_{n=1}^{\infty}(-1)^n/n^2$, we first observe that by Fourier series we may express θ^2 in series form as

$$\theta^2 = \frac{C^2}{3} - \frac{4C^2}{\pi^2}\left(\cos\frac{\pi\theta}{C} - \frac{1}{2^2}\cos\frac{2\pi\theta}{C} + \frac{1}{3^2}\cos\frac{3\pi\theta}{C} - \frac{1}{4^2}\cos\frac{4\pi\theta}{C} + \cdots\right)$$

where C is a constant. Letting $\theta = 0$ and $C = \frac{1}{2}$, we obtain

$$0 = \frac{1}{12} - \frac{1}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

or

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

Accordingly, for $b/a \to \infty$, $\tau_{xz} \to Pa^2/2I$ at $x = y = 0$. That is, the elementary theory of beam also gives the correct result for a very wide beam (Fig. 7-15.1 for $b \gg a$).

Problem Set 7-15

1. Let $b/a = 6$ and $v = 0.3$. For the horizontal line $x = 0$, evaluate the correction factor K [Eq. (7-15.19)] for $y/b = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0.

Review Problems

R-1. Let the resultant vector of the forces acting on the end $z = L$ of a bar be directed along the z axis. Let the resultant moment be zero. Consider the simplest stress distribution that is statically equivalent to the resultant vector and the resultant moment. Hence, by the semi-inverse method solve the problem of the cylindrical bar subjected to a longitudinal end force.

R-2. Let the forces that act on the end of a rod at $z = L$ be statically equivalent to a couple of moment $M_y = M$, where M_y denotes the moment relative to the y axis in the end plane at $z = L$. Compute a statically equivalent system for the end plane $z = 0$. Assume the simplest stress distribution that is statically equivalent to M_y. Hence, solve the problem of bending of a bar subjected to end couple M. Express the stress components, the strain components, and the displacement components in terms of M and material and geometrical properties of the bar.

R-3. Figure R7-3 represents the cross section of a cantilever beam subjected to transverse end load P directed along the x axis. Derive a formula for $f(y)$ to make Ψ vanish on the lateral boundary [see Eq. (7-11.22)]. Discuss the application of the method demonstrated in Sections 7-14 and 7-15 for this problem.

R-4.[9] In Section 7-7 it was assumed that the angle of twist $\theta = \beta z$ was the same for both axes z and z_1. Then it was shown that the stresses, hence the moment M with respect to axis

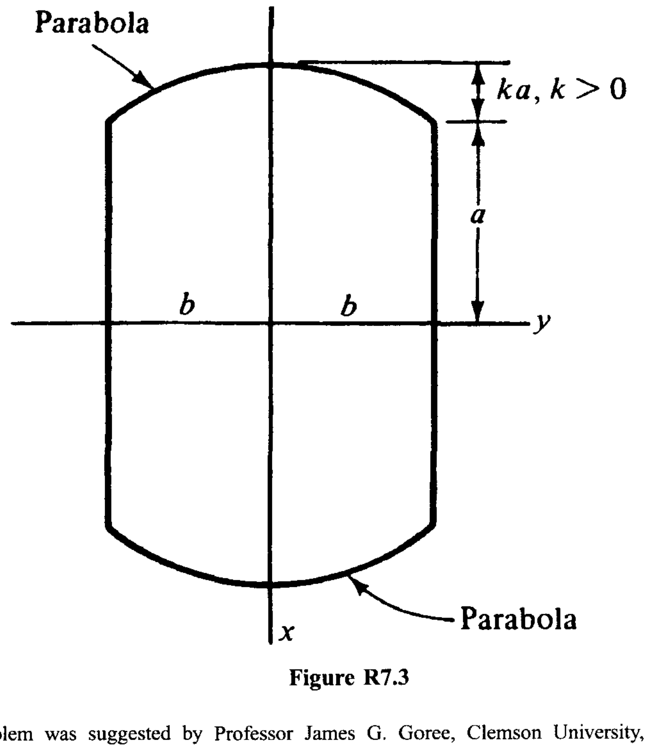


Figure R7.3

[9] This problem was suggested by Professor James G. Goree, Clemson University, Clemson, South Carolina.

z_1, are identical to the stresses and the moment with respect to axis z. Alternatively, we may assume that the twisting moment M for axis z is the same as for axis z_1. Then it may be shown by the equations of elasticity that the twist θ_1 relative to axis z_1 is equal to the twist θ relative to axis z. Verify this statement.

APPENDIX 7A ANALYSIS OF TAPERED BEAMS

Chapter 7 is devoted to prismatic beams. However, for certain structural applications, tapered beams, which have variable moments of inertia to counteract different acting moments, are more efficient than prismatic beams.

As a result of their structural efficiency and suitability for fabrication, web-tapered beams (Fig. 7A-1) are becoming popular in various types of construction. The flexural and torsional behavior of tapered beams has been studied extensively by Lee and his associates (1967, 1972) as well as by Davis et al. (1973). The stability aspects have also been investigated by Kitipornchai and Trahair (1972). As for the shear stresses, Chong et al. (1976) have showed, using principles of mechanics, that the sloping flanges possess vertical components of forces that can either increase or decrease the web shear, depending on the direction of taper and the direction of acting shear.

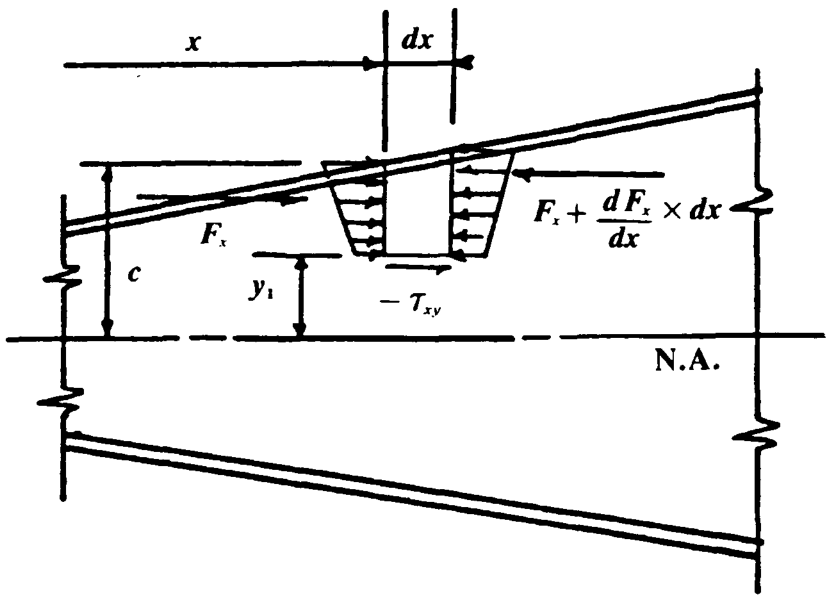


Figure 7A-1 Stresses in tapered beam.

Assumptions of small deflection theory were used. Referring to the stress block in Fig. 7A-1 and summing up forces in the horizontal direction, Chong et al. (1976) found

$$-\tau_{xy} t\, dx = \frac{\partial F_x}{\partial x} dx \tag{7A-1}$$

But

$$F_x = \int_{y_1}^{c} \sigma_x \, dA \tag{7A-2}$$

and

$$\sigma_x = \frac{M_x y}{I_x} \tag{7A-3}$$

and therefore

$$F_x = \frac{M_x}{I_x} \int_{y_1}^{c} y\, dA \tag{7A-4}$$

Let

$$Q_{xy} = \int_{y_1}^{c} y\, dA \tag{7A-5}$$

Then

$$F_x = \frac{M_x Q_{xy}}{I_x} \tag{7A-6}$$

Substituting Eq. (7A-6) into Eq. (7A-1), Chong et al. (1976) found

$$\tau_{xy} = -\frac{1}{t}\frac{\partial}{\partial x}\left(\frac{M_x Q_{xy}}{I_x}\right) \tag{7A-7}$$

Expansion of Eq. (7A-7) yields

$$\tau_{xy} = -\frac{1}{t}\left(\frac{V_x Q_{xy}}{I_x} + \frac{M_x}{I_x}\frac{\partial Q_{xy}}{\partial x} - \frac{M_x Q_{xy}}{I_x^2}\frac{\partial I_x}{\partial x}\right), \tag{7A-8}$$

in which t = thickness of web; F_x = internal force in the x direction; σ_x = normal stress in the x direction: M_x = moment at x: V_x = shear at x; I_x = moment of inertia at x; dA = differential cross-sectional area; and τ_{xy} = shear stress at x.

The first term of Eq. (7A-8) corresponds to the shear stress in beams of constant cross section. The additional terms account for the taper. Equation (7A-7) can be applied to the classical wedge cantilever (Shepherd, 1935), as shown in Fig. 7A-2. The classical solution is

$$\tau_{xy} = -\frac{Py^2}{I_x}\left[\left(\frac{\tan\alpha}{\alpha}\right)^3 \sin^4\theta\right] \qquad \text{(7A-9)}$$

For small tapers

$$\left[\left(\frac{\tan\alpha}{\alpha}\right)^3 \sin^4\theta\right] \to 1 \qquad \text{(7A-10)}$$

For $a = 10°$, the maximum error amounts to 3% if the bracketed term is set equal to unity. Thus, for regular small tapers

$$\tau_{xy} = -\frac{Py^2}{I_x} \qquad \text{(7A-11)}$$

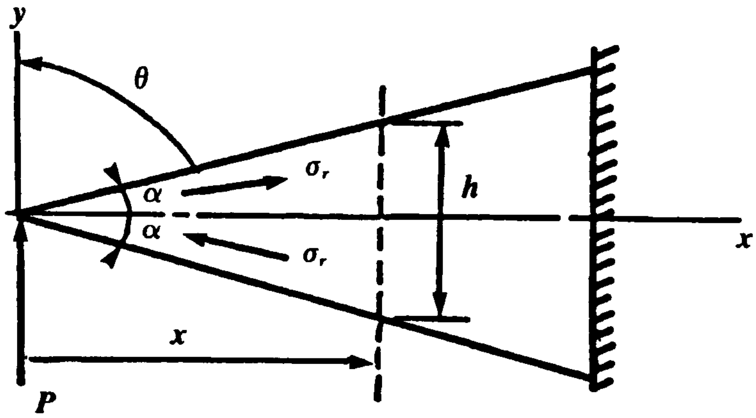


Figure 7A-2 Wedge cantilever loaded at tip

Using Eq. (7A-7), Chong and co-workers obtained

$$M_x = Px \tag{7A-12}$$

$$Q_{xy} = \frac{1}{2}\left[\left(\frac{h}{2}\right)^2 - y^2\right]t; \qquad Q_{xy} = \frac{b}{2}(x^2 \tan^2 \alpha - y^2) \tag{7A-13}$$

in which b = uniform thickness of the wedge. Substitution of Eqs. (7A-12) and (7A-13) into Eq. (7A-7) yields $\tau_{xy} = -Py^2/I_x$, which is identical to Eq. (7A-11).

For shear stresses in tapered beams loaded away from the tip (Fig. 7A-3)

$$I_x = \frac{bh^3}{12} \tag{7A-14}$$

$$Q_{xy} = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \tag{7A-15}$$

Substitution of Eqs. (7A-14) and (7A-15) into (7A-7) gives

$$\tau_{xy} = -\frac{3M_x}{bh^2}\frac{dh}{dx} + \frac{6}{b}\left(\frac{h^2}{4} - y^2\right)\frac{d}{dx}\left(\frac{M}{h^3}\right) \tag{7A-16}$$

The shear distribution of web-tapered beams was investigated using a theory that assumes a radial flexural stress pattern. Finite element analysis and the classical wedge theory were used to check the accuracy of the theory. These independent methods agreed well with each other. (Chong et al., 1976). The presented theory is applicable to wide-flange or box-tapered Hookean beams. Conventionally, the shear–

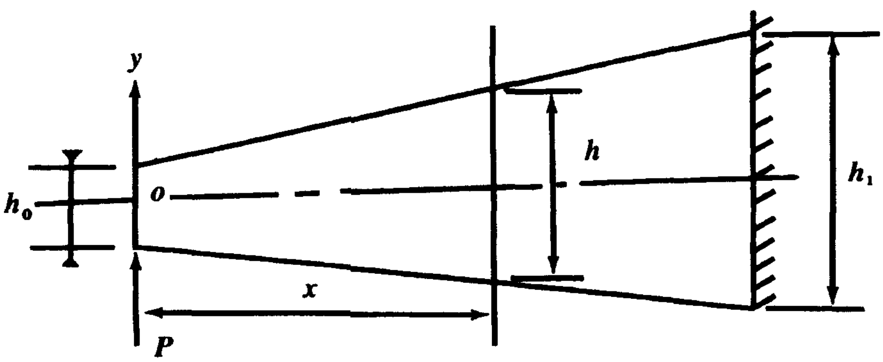


Figure 7A-3 Tapered cantilever beam.

stress distribution is assumed to be uniform with the external shear carried solely by the web. By the proposed theory, significant shears are carried by the flanges, which can be deducted or added to the total web shear.

On the basis of the proposed theory, a simplified analysis procedure was described. We may simply calculate the vertical components of the flange flexural load and subtract or add them from the total vertical shear. The resulting shear was assumed to be carried by the web as a relatively uniform stress distribution (Chong et al., 1976).

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